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MODULE 6

MATHEMATICS 30

Permutations, Combinations, and Statistics

Distance Learning

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Mathematics 30

Module 6

PERMUTATIONS, COMBINATIONS, AND STATISTICS

Distance Learning

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Module 6
Permutations, Combinations, and Statistics
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Welcome

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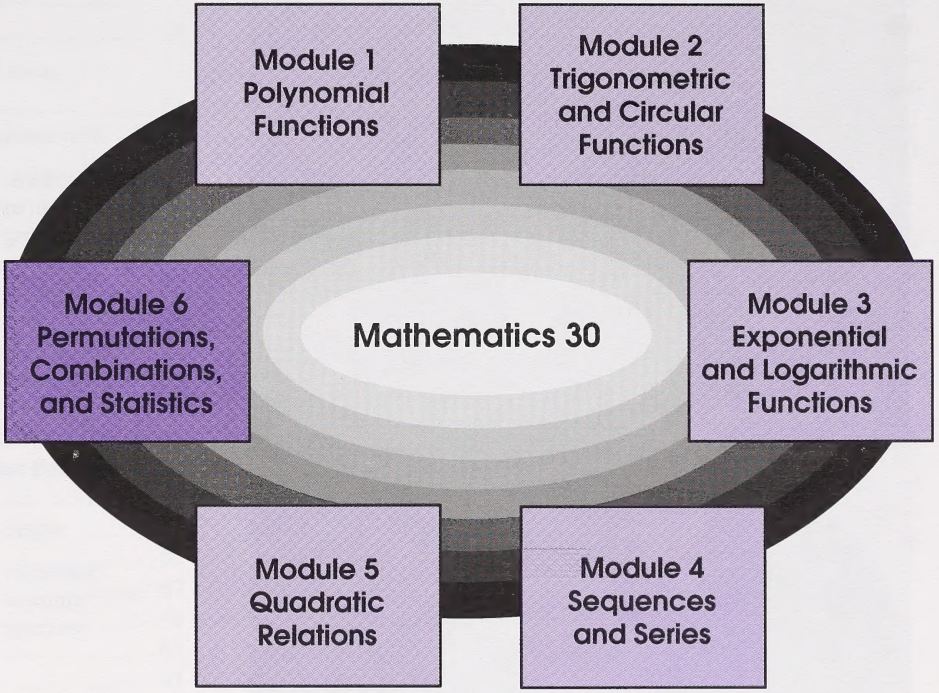
Welcome to Module 6. We hope you'll enjoy your study of Permutations, Combinations, and Statistics.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

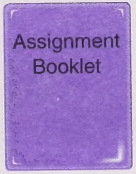
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

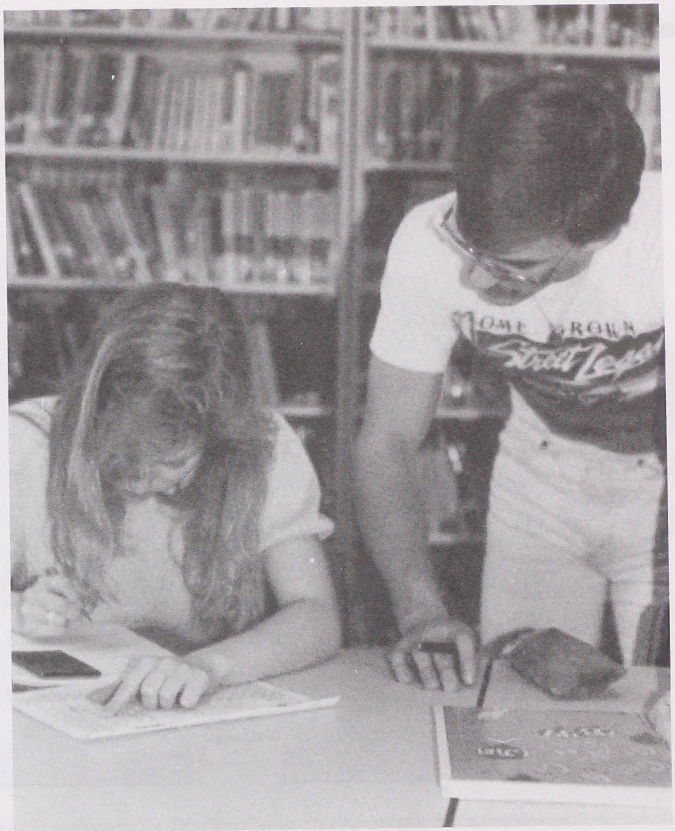
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

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Module Overview

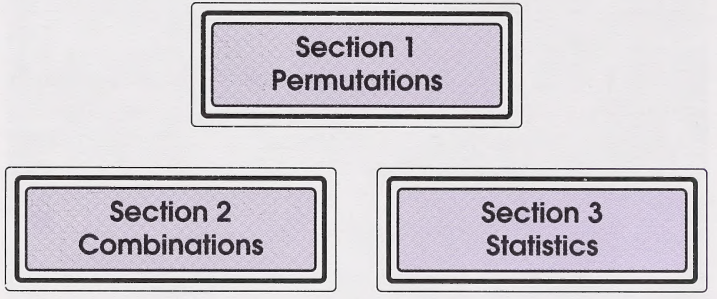
If you have ever forgotten your locker combination, you know how frustrating it can be trying out different arrangements of the numbers you are convinced are right. What is the probability of selecting the correct combination by trial and error alone?

This module focuses on selections and arrangements. Almost every aspect of life involves weighing possibilities and making choices. What books will you read? What courses will you take? What articles of clothing will you wear? In what order will you do the questions on an examination? How will you arrange your schedule?

Section 1 deals with arrangements or permutations—with analysing the number of ways objects can be ordered in a row, or in a circle, or on a ring. In Section 2 you will explore the number of ways objects can be selected. You will also relate selections to the process of expanding powers of binomials.

Section 3 deals with that branch of statistics that looks at how data is distributed. The following are typical of the questions you will address: how height and weight are distributed across the population, what the probability is that you will have to return merchandise before it expires, and what proportion of students are expected to earn 90% or better on a particular examination?

Module 6
Permutations, Combinations, and Statistics



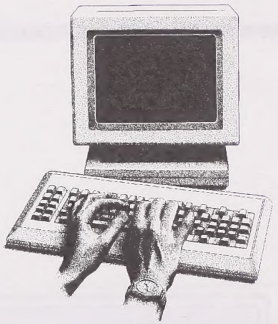
Evaluation

Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete three section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**25 marks**
Section 2 Assignment	**28 marks**
Section 3 Assignment	**22 marks**
Final Module Assignment	**25 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.

If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Permutations

When putting books on a shelf, do you ever wonder how many arrangements are possible? Do you know how many licence plates consisting of three letters followed by three digits can be made? How many ways can a family of five be seated around a table? How many different arrangements are there of three keys on a ring?

These questions are applications of permutations. Permutations are arrangements of objects in a row, in a circle, or on a ring. The study of permutations together with the study of combinations form a branch of mathematics called combinatorics.

In this section you will explore problems in permutations. You will use the fundamental counting principle to determine the number of permutations of a set of objects. In some situations, conditions will be imposed on the way you can arrange those objects. You will develop and apply formulas for linear, circular, and ring arrangements. When some of the objects are identical, you will find out how many permutations are different in appearance. Finally, you will solve probability problems based on permutations.

Activity 1: The Fundamental Counting Principle

There are five candidates for student union chairperson and four for secretary. In how many ways may the positions be filled?

The two positions may be filled in $5 \times 4 = 20$ ways.

In general, if there are r different ways to do one thing and s different ways to do another, there are rs ways to do these two things in succession. This reasoning is called the **fundamental counting principle**.

The fundamental counting principle can be extended when more actions are involved. If the first thing can be done in a ways, the second thing in b ways, the third thing in c ways, and so on, then the number of ways of doing all these things in succession is $a \times b \times c \ldots$ ways. The following example is another way to illustrate this reasoning.

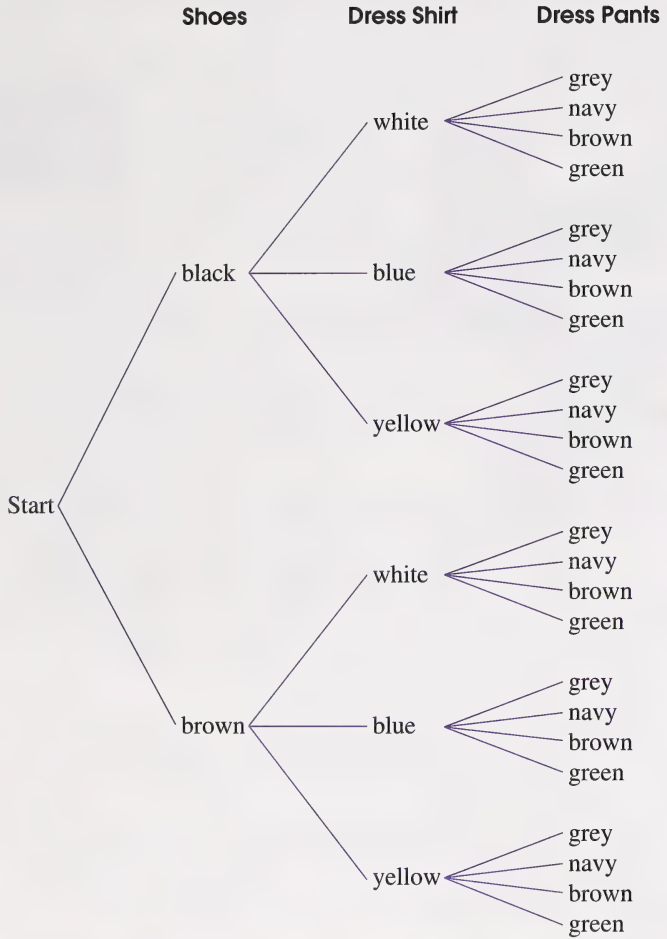
Example 1

Kurt has two pairs of shoes: one black and the other brown. He has three different-coloured dress shirts: white, blue, and yellow. He also has four pairs of different-coloured dress pants: grey, navy, brown, and green. In how many different ways can Kurt dress himself?

Solution

A pair of shoes can be selected in two ways.
A dress shirt can be selected in three ways.
A pair of dress pants can be selected in four ways.

The following tree diagram illustrates the selection process.



By the fundamental counting principle, Kurt can dress himself in $2 \times 3 \times 4 = 24$ ways.

1. An appliance manufacturer builds four different models of ranges; each range is available in any one of three exterior colours. How many different ranges can the manufacturer make?

2. An automobile dealer offers six different models of cars; each model is available in any one of five exterior colours, and with or without air-conditioning. How many different cars could the dealer order?

Check your answers by turning to the Appendix.

The fundamental counting principle applies to a variety of **permutatation** problems.

Examine the following example.

Example 2

Suppose you have a choice of four routes to drive from Edmonton to Red Deer, a choice of three routes from Red Deer to Calgary, and a choice of two routes from Calgary to Lethbridge. How many different routes are there if you want to drive from Edmonton to Lethbridge, via Red Deer and Calgary?

Solution

You can choose one of the four routes from Edmonton to Red Deer. When you get to Red Deer, three routes are available for proceeding to Calgary. Thus, you have $4 \times 3 = 12$ choices of routes from Edmonton to Calgary. For each of these choices, two routes are available for continuing on to Lethbridge. Therefore, there are $12 \times 2 = 24$ different routes from Edmonton to Lethbridge.

Note: The number of routes can be found in one step:

$$4 \times 3 \times 2 = 24$$

3. Using a diagram, show the number of ways of travelling from Town *A* to City *C* via Village *B*, if there are two roads (Highways 14 and 15) connecting Town *A* and Village *B*, and there are four roads (Highways 16, 17, 18, and 19) connecting Village *B* and City *C*.

4. A hockey arena has fifteen gates; seven are on the north side and eight are on the south side.

a. In how many ways can Peter enter by a south gate and leave by a north gate?

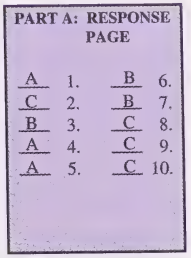
b. In how many ways can Peter enter and leave the stadium?

Check your answers by turning to the Appendix.

The next example shows a different situation.

Example 3

A mathematics test consists of ten multiple-choice questions. Each question has three answers. How many different sets of answers can be chosen?


PART A: RESPONSE PAGE

A	1.	B	6.
C	2.	B	7.
B	3.	C	8.
A	4.	C	9.
A	5.	C	10.

Solution

There are three choices for each question. The total number of choices is as follows:

$$3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^{10}$$
$$= 59\,049$$

A total of 59 049 sets of answers can be chosen.

5. In how many ways can five letters be mailed if there are two mailboxes available?

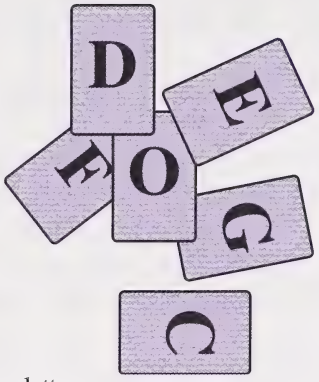
6. There are three mailboxes. Wendy figures out that she can mail her letter in 243 ways. How many letters does Wendy have?

Check your answers by turning to the Appendix.

The next example shows you how to form "words" from a given set of letters. (**Note:** "Words" refers simply to different arrangements of letters.)

Example 4

How many different three-letter words can be formed from the letters C, D, E, F, G, and O if no letter is repeated?



Solution

The first letter can be chosen from any one of the six letters. Having filled this position, the second letter is chosen from the remaining five letters, and consequently, the third letter is chosen from the remaining four letters.

The number of three-letter words is $6 \times 5 \times 4 = 120$.

7. How many four-letter words can be formed using the letters in the word MICE. (**Note:** Letters cannot be repeated.)

8. How many five-letter words can be formed using the letters in the word TRICK if the first letter is C? (Letters cannot be repeated.)

9. How many arrangements of the letters from the word SLUMBER are possible if each arrangement must begin with the letter E, the second letter must not be M, and the fourth letter must be L?

10. How many six-letter words can be formed using the letters from the word NUMBER if the first letter must be a vowel? (Letters cannot be repeated.)

Check your answers by turning to the Appendix.

You can generate different sets of numbers from a set of digits in the number system. You can have numbers with repeating or non-repeating digits. The next example illustrates this.

Example 5

How many different four-digit numerals can be formed using the digits 0, 1, 2, 3, 4, and 5? Assume the digits can be repeated.

Solution

The first digit is not a zero since a number cannot begin with a zero.

The next three digits can repeat since there is no restriction.

The four-digit number is represented by four boxes.

First Digit	Second Digit	Third Digit	Fourth Digit
5	6	6	6

There are five choices for the first digit since 0 cannot be used. Each of the other digits have six choices since the digits can be repeated.

The total number of four-digit numerals is $5 \times 6 \times 6 \times 6 = 1080$.

11. How many three-digit numbers less than 400 can be formed from the digits 2, 3, 5, 7, and 8? (Digits can be repeated.) Solve this problem by filling in the blanks.

a. Can the first digit be 5, 7, or 8? _____

b. The first digit can be chosen in _____ ways.

c. The second digit can be chosen in _____ ways.

d. The third digit can be chosen in _____ ways.

e. Apply the fundamental counting principle. The number of three-digit numbers is _____.

12. How many three-digit odd numbers between 200 and 600 can be formed from the digits 1, 2, 4, 7, and 9? (Digits cannot be repeated.)

13. How many different licence plates can be made using three letters followed by three digits from the digits 1, 2, 3,..., 8, 9? (Repetition of letters and digits is not allowed.)

Check your answers by turning to the Appendix.

A task can be performed in a many ways. Objects can be arranged in a variety of ways. You can use the fundamental counting principle to determine the number of ways actions can be accomplished.

Activity 2: Defining and Using ${}_nP_n$ and $n!$

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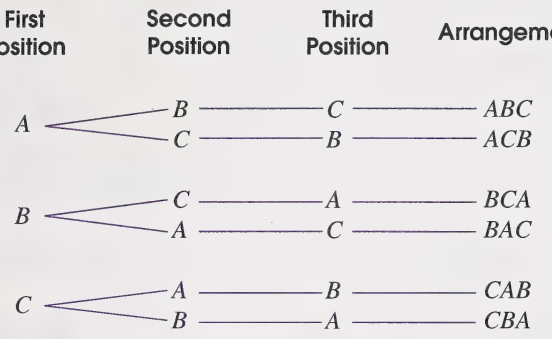
Grain! Lumber! Chemicals! Trains carry a wide assortment of goods across the country. How many different ways could the cars in a train be arranged? Do the cars have to be arranged in any special order?

In the previous activity, the fundamental counting principle was used to find the number of ways in which actions can be performed. In this activity, more attention will be paid to the arrangements of objects where order must be considered.

An arrangement of things along a straight line in a definite order is called a **linear permutation**.

Consider the following situation.

Three boxcars, A, B, and C, of a train set are lined up. How many different ways can the boxcars be arranged in a line? The following tree diagram shows the possible arrangements.



From the tree diagram, you can see that A, B, and C can be arranged in six ways.

You can also solve this problem by using the fundamental counting principle.

By the fundamental counting principle, the total number of arrangments of the three boxcars is $3 \times 2 \times 1 = 6$.

Look at the following example.

Example 1

How many different seven-digit numbers can be made from the digits 1 through 7 with no repetition?

Solution

Let the number be represented by seven boxes.

The first digit could be any of the seven digits, the second digit could be any one of the six remaining digits, and so on. This could be shown by filling in the numbers in the boxes.

Therefore, $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ numbers can be made from these seven digits.

Example 1 represents a permutation of seven digits taken seven at a time. There is a special symbol used to designate the number of permutations.

The symbol used to represent the number of permutations of n things taken n at a time is ${}_nP_n$.

In Example 1, there are seven digits taken all seven at a time. The symbol used to represent the number of permutations is ${}_7P_7$.

$$
\begin{aligned}
{}_7P_7 &= 7\times6\times5\times4\times3\times2\times1 \\
&= 5040
\end{aligned}
$$

If there are n objects, the number of permutations of all of the objects will be ${}_nP_n = (n)(n-1)(n-2)(n-3)\ldots3\times2\times1$.

In **combinatorics**, multiplication of consecutive numbers is common. In order to avoid writing all the numbers, a new notation $\boldsymbol{n!}$ is used.

Combinatorics is the study of permutations and combinations.

$\boldsymbol{n!}$ is read as ***n* factorial** and it represents the product of the **first** n natural numbers.

For example, $1! = 1$

$2! = 2\times1$

$3! = 3\times2\times1$

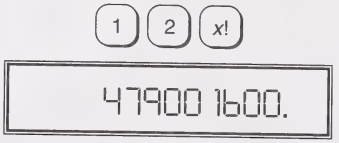
$4! = 4\times3\times2\times1$

.....

$n! = n(n-1)(n-2)\ldots3\times2\times1$

In Example 1, ${}_7P_7 = 7!$.

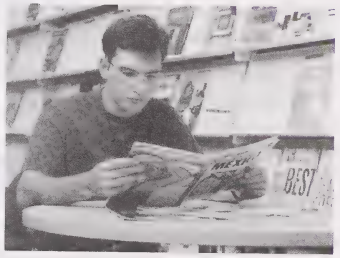
How do you calculate 12!? It means a lot of multiplications. A scientific calculator can make things much easier for you. Use a calculator to calculate 12!.



Note: Some calculators may use $n!$ rather than $x!$.

1. In how many ways can the letters in the word ALBERT be arranged?

2. In how many different orders can Jerritt read five different magazines?



Check your answers by turning to the Appendix.

It is important that you know how to simplify factorials. Look at the following examples.

Example 2

Simplify $\frac{9!}{4!}$.

Solution

$$\frac{9!}{4!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{4!}$$
$$= \frac{9\times8\times7\times6\times5\times\cancel{4!}}{\cancel{4!}}$$
$$= 9\times8\times7\times6\times5$$
$$= 15\,120$$

Factor the numerator to the $4!$ stage since the denominator is $4!$. Cancel the $4!$ in the numerator with the $4!$ in the denominator.

3. Find the value of $\frac{10!}{7!2!}$.

4. Find the value of $\frac{4!}{5!2!} + \frac{6!}{3!2!}$.

Check your answers by turning to the Appendix.

Note that $n!$ can be written in a variety of equivalent forms.

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)!$$
$$= n(n-1)(n-2)(n-3)!$$
$$\vdots$$

When a factorial is denoted by a variable, it is necessary to change it to one of these equivalent forms so that you can reduce a rational expression with factorials in the numerator and denominator. Look at the following examples.

Example 3

Simplify $\frac{n!}{(n-3)!}$.

Solution

$$\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}}$$
$$= n(n-1)(n-2)$$
$$= n^3 - 3n^2 + 2n$$

Example 4

Simplify $\frac{(n-2r+3)!}{(n-2r)!}$.

Solution

$$\frac{(n-2r+3)!}{(n-2r)!} = \frac{(n-2r+3)(n-2r+2)(n-2r+1)\cancel{(n-2r)!}}{\cancel{(n-2r)!}}$$
$$= (n-2r+3)(n-2r+2)(n-2r+1)$$

5. Express $(k-2)(k-3)!$ as a single factorial.

6. Simplify $\dfrac{(n+r-2)!}{(n+r-5)!}$.

7. Express $(m+3)(m+2)!$ as a single factorial.

8. Solve for n.

a. $\dfrac{(n-3)!}{(n-4)!} = 7$ **b.** $\dfrac{(n-1)!}{(n+1)!} = \dfrac{1}{6}$

9. Reduce the following:

a. $\dfrac{(a+2)!}{a(a+2)}$ **b.** $\dfrac{(n-r+2)!}{(n-r-2)!}$ **c.** $\dfrac{(a-1)!}{(a-3)!\,2!}$

Check your answers by turning to the Appendix.

Some word problems dealing with arrangements are not that straightforward. Look at the next example.

Example 5

A bookshelf contains five different algebra books and seven different physics books. How many different ways can these books be arranged if the algebra books are to be kept together?

Solution

There are twelve books on the shelf.
The five algebra books can be arranged in $5!$ ways. The algebra books are considered as one unit, so there are eight items (1 algebra unit $+7$ physics books) to be arranged. The number of arrangements is ${}_8P_8$.

$$\therefore \text{ number of arrangements} = 5! \times {}_8P_8$$
$$= 5! \times 8!$$
$$= 120 \times 40\,320$$
$$= 4\,838\,400$$

The books can be arranged in 4 838 400 ways.

10. How many different signals can be made from four different flags if each signal is to consist of four flags hung in a row?

11. In how many ways can four chemistry books and three biology books be placed on a shelf if the biology books are kept together?

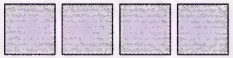
12. In how many ways can six algebra books and four geometry books be placed on a shelf if the geometry books are kept together on either end of the shelf?

Check your answers by turning to the Appendix.

This activity emphasized that a permutation is an arrangement in which order is important.

Activity 3: Defining and Using ${}_nP_r$ to Solve Linear Permutations

It was defined in Activity 2 that ${}_nP_n = n!$ is the number of linear permutations of n objects taken n at a time. What is the number of linear permutations of n objects taken r at a time? For example, how many different four-digit numbers can be made from the digits 1 through 7 with no repetitions? In Activity 2: Example 1, seven boxes were used to house the seven digits taken at a time. Now the number to be made has four digits. Only four boxes are needed.



The first digit could be any one of seven digits, the second digit could be any one of the six remaining digits, the third digit could be any one of the five remaining digits, and the fourth digit could be any one of the four remaining digits.

By the fundamental counting principle, the total number of four-digit numbers is $7 \times 6 \times 5 \times 4 = 840$.

In general, if there are n objects taken r at a time, then the first object must be selected from a total of n objects, the second object must be selected from $(n-1)$ remaining objects, the third one must be selected from $(n-2)$ remaining objects, and the nth object must be selected from $(n-r+1)$ remaining objects.

The symbol used to denote the number of permutations of n objects taken r at a time is ${}_nP_r$.

Therefore, ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$.

Since $n(n-1)(n-2)\ldots(n-r+1)(n-r)! = n!$, then ${}_nP_r\,(n-r)! = n!$.

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

According to this formula, ${}_nP_n = \frac{n!}{(n-n)!}$

$$= \frac{n!}{0!}$$

In Activity 2, ${}_nP_n$ is defined as $n!$. For this formula ${}_nP_r = \frac{n!}{(n-r)!}$ to hold for $r = n$, $0!$ must be defined as 1 so that ${}_nP_n = \frac{n!}{0!} = \frac{n!}{1!} = n!$. Since $1! = 1$, $0! = 1! = 1$.

Note: $n \geq r$

Look at the following examples.

Example 1

At a parking lot, five cars are to be parked side by side along the wall. If there are seven cars, how many different ways can this be done?

Solution

Cars are to be parked side by side along the wall; therefore, it is a linear permutation problem.

This problem can be solved by using the formula ${}_nP_r = \frac{n!}{(n-r)!}$, where $n = 7$ and $r = 5$.

$${}_7P_5 = \frac{7!}{(7-5)!}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2!}}{\cancel{2!}}$$

$$= 7 \times 6 \times 5 \times 4 \times 3$$

$$= 2520$$

The cars can be parked in 2520 ways.

In permutations you may have to add, subtract, or multiply arrangements. It is important to know how to do these operations.

Example 2

Evaluate ${}_3P_1 \times {}_6P_4$.

Solution

$${}_3P_1 \times {}_6P_4 = \frac{3!}{(3-1)!} \times \frac{6!}{(6-4)!}$$

$$= \frac{3!}{2!} \times \frac{6!}{2!}$$

$$= (3) \times (6 \times 5 \times 4 \times 3)$$

$$= 1080$$

Example 3

Evaluate ${}_5P_4 + {}_7P_1 - {}_3P_2$.

Solution

$$
\begin{aligned}
{}_5P_4 + {}_7P_1 - {}_3P_2 &= \frac{5!}{(5-4)!} + \frac{7!}{(7-1)!} - \frac{3!}{(3-2)!} \\
&= 120 + 7 - 6 \\
&= 121
\end{aligned}
$$

1. Evaluate the following:

a. ${}_4P_3 + {}_5P_2$

b. $\dfrac{7!}{5!0!}$

c. ${}_{10}P_3 \times {}_5P_5$

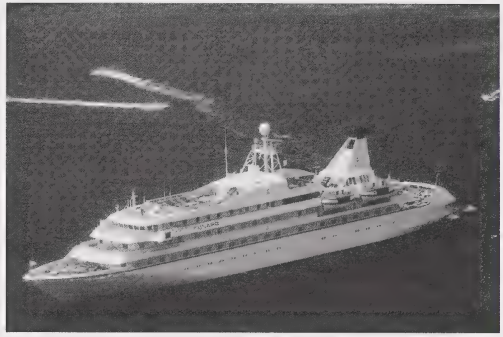
Check your answers by turning to the Appendix.

Some problems in permutations can be solved by using the form ${}_nP_r$ repeatedly for different possibilities; and then adding the results of those possibilities.

Study the following example.

Example 4

When radio contact is lost, ships use a combination of six different signal flags different to send different messages. If no more than four flags can be used at any one time, how many different signals can be sent? (No flags is not considered a signal.)



Solution

If you choose any four flags, the number of signals is ${}_6P_4$.

If you choose any three flags, the number of signals is ${}_6P_3$.

If you choose any two flags, the number of signals is ${}_6P_2$.

If you choose any one flag, the number of signals is ${}_6P_1$.

$$\therefore \text{ number of signals} = {}_6P_4 + {}_6P_3 + {}_6P_2 + {}_6P_1$$

$$= \frac{6!}{(6-4)!} + \frac{6!}{(6-3)!} + \frac{6!}{(6-2)!} + \frac{6!}{(6-1)!}$$

$$= \frac{6!}{2!} + \frac{6!}{3!} + \frac{6!}{4!} + \frac{6!}{5!}$$

$$= (6\times5\times4\times3) + (6\times5\times4) + (6\times5) + (6)$$

$$= 360 + 120 + 30 + 6$$

$$= 516$$

A total of 516 signals can be sent.

2. How many different numbers can be formed from the digits 1, 2, 3, 4, and 5 if no more than four digits can be used and no repetition is allowed?

3. In how many ways can ten different pictures be hung in a row if no more than five are to be hung? (At least one picture is hung.)

4. Determine the number of different words of six or more letters each, that can be formed with the letters of the word TERMINAL. Each letter is used only once.

Check your answers by turning to the Appendix.

You can apply the formula ${}_nP_r = \frac{n!}{(n-r)!}$ to solve equations. This is shown in the next example.

Example 5

Determine the value of n which makes the following statement true.

$${}_nP_6 = 8\times{}_{n-1}P_5$$

Solution

$${}_nP_6 = \frac{n!}{(n-6)!}$$

$${}_{n-1}P_5 = \frac{(n-1)!}{(n-1-5)!}$$

$$= \frac{(n-1)!}{(n-6)!}$$

If $\frac{n!}{(n-6)!} = 8\times\frac{(n-1)!}{(n-6)!}$, then $n! = 8\times(n-1)!$.

$$\frac{n!}{(n-1)!} = 8$$

$$\frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = 8$$

$$n = 8$$

5. Determine the values of n which make the following statements true.

a. $\left(\frac{1}{2}\right){}_nP_7 = {}_{n-2}P_5$, where $n \in N$

b. $4\times {}_nP_3 = 6\times {}_{n-1}P_3$

6. Prove ${}_nP_{r+2} = (n-r)(n-r-1)\times {}_nP_r$.

Check your answers by turning to the Appendix.

Look at the following example of a linear permutation.

Example 6

There are two girls, five boys, and five chairs in a row. It is required that five persons be seated in a row such that the two girls occupy the first two seats. How many different ways can this be done?

Solution

The two girls are considered as one unit. There are ${}_2P_2 = 2$ ways to seat these two girls.

The boys may be seated in ${}_5P_3$ ways because there are three chairs for five boys.

$$\therefore \text{ number of ways} = {}_2P_2 \times {}_5P_3$$
$$= 2\times\frac{5!}{(5-3)!}$$
$$= 2\times(5\times4\times3)$$
$$= 120$$

The children may be seated in 120 ways.

Sometimes objects (including digits) can be used more than once in an arrangement. In this case, the fundamental counting principle can provide the best explanation.

7. Five men and four women are seated in a row such that the women occupy the even places. How many such arrangements are possible?

8. In how many ways can three adults and four children line up for a photo, if the children must stand together?

9. How many serial numbers can be formed using two different letters followed by four different digits?

Check your answers by turning to the Appendix.

Some permutations involve restrictions. When restrictions are placed on the ways in which objects are to be arranged, deal with the restrictions first.

The next two examples are problems on restrictions.

Example 7

How many different five-digit credit card numbers can be formed if the first digit cannot be zero?

Solution

There are ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
The first digit cannot be zero. Therefore, there are nine ways to choose the first digit.
The other digits can be any one of the ten digits.
If five boxes are used to house these five digits, the first box can be filled by any one of the nine digits.
There are ten ways to fill each of the other four boxes.

9	10	10	10	10

The number of five-digit numbers is $9 \times 10 \times 10 \times 10 \times 10 = 90\,000$.

The next example has more restrictions:

Example 8

How many six-digit odd numbers less than 500 000 can be formed from the digits 3, 4, 5, 6, 7, and 8?

Solution

The first digit can be 3 or 4.
The last digit must be an odd number (3, 5, and 7).
The other digits can be any one of the six digits.

Use six boxes to house the numbers.

2	6	6	6	6	3

The number of arrangments is $2 \times 6 \times 6 \times 6 \times 6 \times 3 = 7776$.

10. How many numbers between 200 and 500 can be formed by using the digits 1, 2, 3, 4, 5, and 6 if no digit is repeated in any number?

11. How many telephone numbers can be formed if the number is to be ten digits in length. The first three digits must be 403; the fourth digit can be any number except for 0 or 1; and the remaining six digits can be any one of the ten digits (0 to 9) but do not repeat.

12. Miss Singh's classroom has five rows of five desks each. She has 20 students. The 5 shortest students will be seated in the front row. The remaining 15 students can sit anywhere in the four rows. In how many ways can the students be assigned to the desks? (Leave your answer in factorial form.)

13. A teacher has two different books and seven students.

a. In how many ways can these two books be given to the students if both books may be given to the same student.

b. In how many ways can these two books be given to the students if both books may not be given to the same student?

14. If no two marbles and no two beads are to be placed next to each other, how many different arrangements can be made by four different marbles and four different colour beads in a row?

15. A student has two different biology books, three different chemistry books, and five different physics books. In how many ways can these books be arranged on a bookshelf so that the biology books stand together, the chemistry books stand together, and the physics books stand together?

Check your answers by turning to the Appendix.

If some of the objects you want to arrange are identical, how do you determine the number of permutations? For example, how many arrangements can be formed by using the letters of the word SEE?

If the letters were different, there would be $3! = 6$ ways to arrange them. The problem is that the two Es are really identical and indistinguishable. If you pretend that the two Es are different and use a capital letter E and a small letter e to denote the two of them, the possible arrangements are as follows:

SEe	SeE
ESe	eSE
EeS	eES

Since the Es are really identical, there are actually three different permutations: SEe, ESe, and EeS. For every position of the letter S, there are $2! = 2$ ways to arrange the Es. Therefore, the total number of arrangments is $\frac{3!}{2!} = \frac{3\times2\times1}{2\times1} = 3$ ways.

In general, the number of permutations P of n objects taken all at a time, of which u are alike, v are the same of another kind, w are the same of another kind, and so on, is as follows:

$$P = \frac{n!}{u!\,v!\,w!\ldots}, \text{ where } u+v+w+\ldots = n$$

Look at the following example.

Example 9

How many distinguishable permutations can be made from all the letters of the word REFERENCE?

Solution

There are two Rs and four Es.

$$\therefore \text{ number of permutations} = \frac{9!}{2!4!1!1!1!}$$

$$= \frac{9\times8\times7\times6\times5\times\cancel{4!}}{2!\cancel{4!}}$$

$$= \frac{9\times8\times7\times6\times5}{2\times1}$$

$$= 7560$$

A total of 7560 permutations can be made.

Sometimes the objects do not have to be identical, but you have to treat them as if they were alike.

Example 10

How many distinguishable permutations can be made from all the letters of the word REACH if the consonants must be in alphabetical order?

Solution

If the order of the consonants cannot be changed, they can be treated as if they were alike. The three consonants are C, H, and R.

$$P = \frac{5!}{3!1!1!}$$

$$= 5\times4$$

$$= 20$$

Therefore, 20 permutations can be made.

16. How many distinguishable permutations can be made from all the letters of the word PEERLESS?

17. A teacher wants to distribute three copies of a chemistry book and five copies of a physics book to eight students, and each student is to receive one book. In how many different ways can this be done?

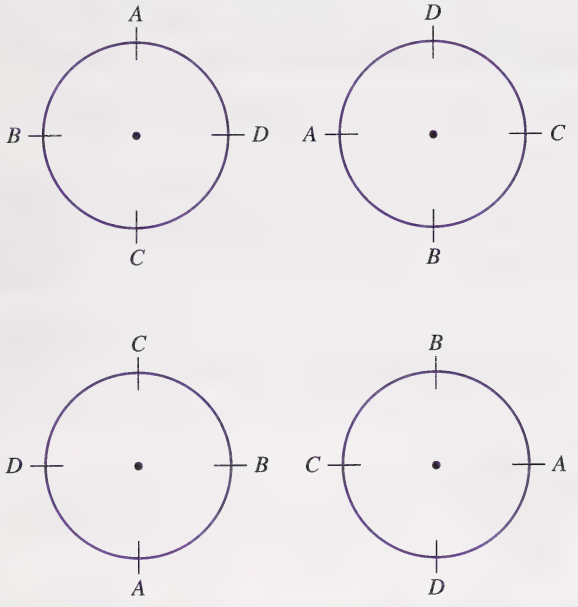
18. How many distinct permutations can be formed from all the letters of the word THERMOMETER?

Check your answers by turning to the Appendix.

Sometimes, it is easier to solve permutation problems by reasoning. Now you can calculate ${}_nP_r$ using calculators.

Activity 4: Circular Permutations and Permutations with Reflections

In the previous activity, you studied linear permutations. In this activity, you are going to study **circular permutations**. A circular permutation is an arrangement of objects around a circle. Imagine a merry-go-round of four persons (A, B, C, and D) equally spaced. The four illustrated arrangements are equivalent.



A different arrangement is made only when the position of a person relative to another one is changed. In order to find the number of distinguishable arrangements, a person is assigned a position in the circle. Then the number of distinguishable arrangements is the number of arrangements of the remaining $(4-1)=3$ persons as if they were in a straight line. For this merry-go-round of four persons, there are $3!$ arrangements.

In general, the number of ways of arranging n objects around a circle is **$(n-1)!$ ways**.

Look at the following example.

Example 1

In how many different ways can five people be arranged in a circle?

Solution

This is a circular permutation problem. You have to fix the position of one of the five people.

Let one person be seated anywhere.

$$\therefore \text{ number of ways} = (5-1)!$$
$$= 4!$$
$$= 24$$

The people can be arranged in 24 ways.

1. In how many ways can nine folk dancers form a circle?

2. In how many ways can a football team of twelve players form a circular huddle?

3. A gardener wishes to plant ten different flowers in a circular arrangement. In how many ways can this be done?

4. How many ways are there of arranging the numerals $0, 1, 2, \ldots, 36$ on a roulette wheel? (Leave your answer in factorial form.)

Check your answers by turning to the Appendix.

Example 2

In how many ways can seven people be seated at a round table if three of them must always sit together?

Solution

The three who must sit together can be arranged in $3!$ ways.

Consider the group of three as one person. Therefore, the remaining four people plus the group of three make up five people.

Five people can be seated at a round table in $(5-1)!$ ways.

$$\therefore \text{ number of ways} = 3! \times (5-1)!$$
$$= (3\times 2\times 1)(4\times 3\times 2\times 1)$$
$$= 6\times 24$$
$$= 144$$

The seven people can be arranged in 144 ways.

Example 3

In how many ways can the digits 1, 2, 3, 4, 5, 6, 7, and 8 be placed in a circle if each even digit is to be between two odd digits?

Solution

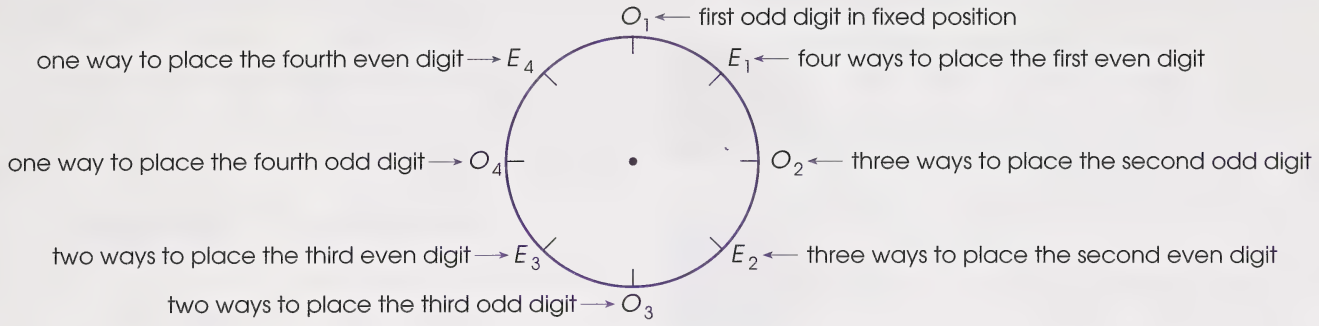
Assume that the odd digits are to be placed first. One position is assigned to an odd digit, and the other three odd digits can be arranged in $3!$ ways.

Now there are four positions for the four even digits. They can be arranged in $4!$ ways.

$$\therefore \text{ number of circular permutations} = 3! \times 4!$$
$$= 144$$

There are 144 ways to arrange the digits.

Look at Example 3 from a different perspective. In the following diagram, let O represent odd digits and E represent even digits.


O_1 ← first odd digit in fixed position
one way to place the fourth even digit → E_4
E_1 ← four ways to place the first even digit
one way to place the fourth odd digit → O_4
O_2 ← three ways to place the second odd digit
two ways to place the third even digit → E_3
E_2 ← three ways to place the second even digit
two ways to place the third odd digit → O_3

The total number of circular permutations is $1\times4\times3\times3\times2\times2\times1\times1=144$ ways.

5. A florist wants to use ten different flowers and two identical roses to make a circular arrangement. In how many ways can this be done?

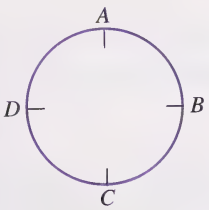
6. Five boys and five girls sit at a round table. Boys and girls must alternate. In how many different ways can this be done?

7. Seven different marbles are placed in a circle. How many different arrangements can be made if three of them must be placed in a position adjacent to each other?

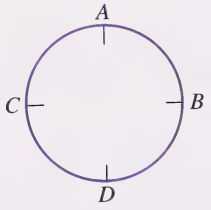
Check your answers by turning to the Appendix.

You have just learned that n objects can be arranged in a circle in $(n-1)!$ ways. Consider the example of arranging four objects (label them A, B, C, and D) in a circle. From your previous work, you know that four objects can be arranged in a circle in $(4-1)!$ or six ways. The six arrangements are shown as follows:

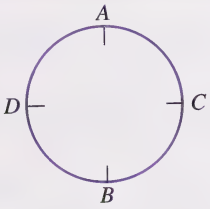
Arrangement 1



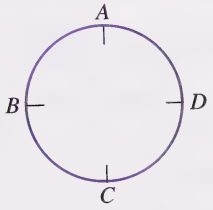
Arrangement 2



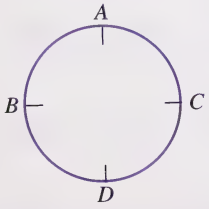
Arrangement 3



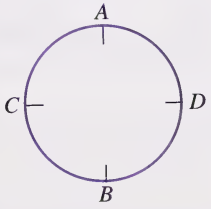
Arrangement 4



Arrangement 5

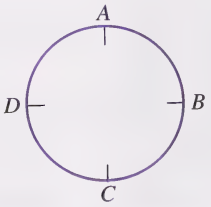


Arrangement 6

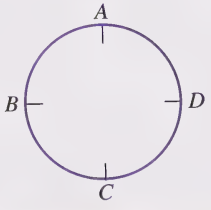


If these arrangements can be turned over as in the case of a necklace or a key chain, then Arrangements 1 and 4 are identical since Arrangement 4 can be obtained by flipping the ring in Arrangement 1 about a vertical line AC, without repositioning the objects on the ring. Arrangement 4 is also the mirror reflection of Arrangement 1.

Arrangement 1



Arrangement 4



Similarly, Arrangements 2 and 5 are identical if you are dealing with a ring that can be turned over. Also, Arrangements 3 and 6 are identical. Therefore, four objects can be arranged in a ring (that can be turned over) in only three ways. In other words, the number of arrangements is half as many as before.

The kind of arrangement like a necklace or a key chain is called a circular permutation with reflections or a **ring permutation**.

The number of ways of arranging n objects in a ring that can be turned over is as follows:

$$\frac{(n-1)!}{2}, \text{ where } n > 2$$

Study the following examples.

Example 4

In how many ways can nine different charms on a charm bracelet be arranged?

Solution

$$\text{number of ways} = \frac{(n-1)!}{2}$$
$$= \frac{(9-1)!}{2}$$
$$= \frac{8!}{2}$$
$$= 20\,160$$

Therefore, the charms can be arranged in 20 160 ways.

Example 5

A key chain has seven different keys. If two of them are attached together, how many different arrangements can be made?

Solution

The two keys together are considered as one group and can be arranged in $2! = 2$ ways.

The remaining five keys and this one group can be arranged in $\frac{(6-1)!}{2}$ ways.

$$\therefore \text{ number of arrangements} = 2 \times \frac{(6-1)!}{2}$$
$$= 5!$$
$$= 120$$

The keys can be arranged in 120 ways.

8. In how many ways can five keys be arranged on a key chain?

9. How many different ways can seven different charms be arranged on a bracelet?

10. Two different car keys, one house key, and an identification tag are placed on a key chain. How many arrangements are possible?

11. Three different cone-shaped jewels and twenty-five different beads are arranged on a necklace so that the jewels are kept together. How many arrangements are possible? (Leave your answer in factorial form).

Check your answers by turning to the Appendix.

In a circular permutation, one position is fixed. In a circular permutation with reflections (ring permutation), the arrangements are half of the results obtained in a circular permutation.

Activity 5: Probability Using Permutations

You studied probability in Mathematics 20.

$$\text{probability} = \frac{\text{number of favourable cases}}{\text{number of possible cases}}$$

$$P(E) = \frac{n(A)}{n(S)}$$

A = favourable cases
S = possible cases
n = number
$P(E)$ = probability

Many questions in probability involve permutations. If the number of possible cases represents the number of arrangements, you have to use permutations to determine the number of cases. Look at the following examples.

Example 1

Ten people line up at a theatre. They are of different heights. What is the probability that they form a queue in descending order of height?

Solution

If the ten people line up randomly, then the number of possible queues is the number of arrangements of the ten people.

$${}_{10}P_{10} = 10!$$

Only one queue has all ten people in descending order of height; therefore, the number of favourable cases is one.

$$\therefore P(E) = \frac{1}{10!}$$

$$= \frac{1}{3\,628\,800}$$

The probability that the ten people line up in descending order of height is $\frac{1}{3\,628\,800}$.

Example 2

There are twelve bicycles in a race. Horst, Paul, and Kathy are members of the same team. What is the probability that they would finish in the top three?

Solution

The total possible cases consist of any group of three persons to finish in the top three.

$$\therefore n(S) = {}_{12}P_3$$

$$= \frac{12!}{(12-3)!}$$

$$= \frac{12!}{9!}$$

$$= 12 \times 11 \times 10$$

$$= 1320$$

The number of favourable cases is the number of cases that Horst, Paul, and Kathy would finish in the top three.

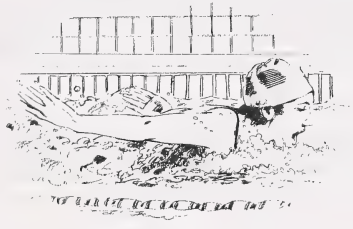
$$\therefore n(A) = {}_3P_3$$
$$= 3!$$
$$= 6$$

$$P(E) = \frac{6}{1320}$$
$$= \frac{1}{220}$$

So, the probability that this team finishes in the top three is $\frac{1}{220}$.

1. Each number from 1 to 7 is written on a piece of paper and placed in a box. All seven numbers are drawn at random from the box one at a time. What is the probability that the numbers drawn are in descending order?

2. There are eight swimmers in a competition. *A*, *B*, and *C* are three of the swimmers. What is the probability that *A*, *B*, and *C* will finish in the top three?



3. Three cards are drawn, one after the other randomly from an ordinary deck of 52 cards without replacement. What is the probability that the first card is a black king, the second card is a red queen, and the third card is a black jack?

4. There are three red balls and seven blue balls in a bag. Three balls are picked at random.

a. If each ball is replaced before the next one is picked, what is the probability that at least two of the three picked balls are red?

b. If no ball is replaced before the next one is picked, what is the probability that at least two of the three picked balls are red?

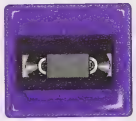
Check your answers by turning to the Appendix.

In some experiments, it is very difficult, and also impractical, to list all the results because of its size. It is convenient to use permutations to find the number of favourable cases and the number of possible cases.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help



You may find it helpful to view the video titled *Permutations and Combinations*, from the *Catch 30* series, ACCESS Network. This program will reinforce the concepts on permutations. You may view the second part of the program on combinations at the beginning of Section 2 or in the Extra Help part of Section 2. This video is available from the Learning Resources Distributing Centre.

If you can do one thing in five different ways and another thing in seven different ways, you can do these two things in succession in $5 \times 7 = 35$ ways. This reasoning is called the fundamental counting principle. Some books call it the multiplicative counting principle.

Example 1

A committee has six members. In how many ways can they choose a chairperson and a secretary?

Solution

Draw two boxes: one represents the number of ways to choose a chairperson; the second represents the number of ways to choose a secretary.

There are six different ways to choose a chairperson. Write 6 in the first box.

There are five persons left; therefore, there are five ways to choose a secretary. Write 5 in the second box.

There are $6 \times 5 = 30$ ways to choose a chairperson and a secretary.

1. In how many ways can a president, secretary, and a treasurer be chosen from a committee of ten people?

Check your answer by turning to the Appendix.

Arranging things in a row is called linear arrangement or linear permutation. If you arrange five different objects in a row, the number of arrangements is $5 \times 4 \times 3 \times 2 \times 1$, and the product of the first five natural numbers can be denoted by the symbol $5!$ (read as **five factorial**). If n different objects are arranged in a row, the number of arrangements is $n!$.

You have to know how to expand a factorial. Examine the next example.

Example 2

Expand $(n-r-3)!$.

Solution

When you expand $(n-r-3)!$, the first factor is $(n-r-3)$. Each of the factors that follow are reduced by 1. The last factor is 1.

$$(n-r-3)!=(n-r-3)(n-r-4)(n-r-5)\ldots 2\times 1$$

2. Expand $(n+r-4)!$.

3. Give the factorial form for the factors $(p-q+6)(p-q+5)(p-q+4)\ldots 2\times 1$.

Check your answers by turning to the Appendix.

If there are eight objects and you can choose any four of them to make arrangements, the number of arrangements is $8\times 7\times 6\times 5$ and is denoted by ${}_8P_4$.

The symbol ${}_8P_4$ can also be defined as follows:

$$\begin{aligned} {}_8P_4 &= \frac{8!}{(8-4)!} \\ &= \frac{8!}{4!} \\ &= \frac{8\times 7\times 6\times 5\times \cancel{4!}}{\cancel{4!}} \\ &= 8\times 7\times 6\times 5 \\ &= 1680 \end{aligned}$$

Factor the numerator until you get to the same factorial as the denominator and cancel.

In general, ${}_nP_r = \frac{n!}{(n-r)!}$.

Some problems involve different kinds of permutations. The total number of permutations is the sum of the different permutations.

Example 3

How many different words can be formed using any number of letters in the word HOUR? No letter is repeated.

Solution

If one letter is used to form a word, calculate ${}_4P_1$.

$$
\begin{aligned}
{}_4P_1 &= \frac{4!}{(4-1)!} \\
&= \frac{4!}{3!} \\
&= \frac{4 \times 3!}{3!} \\
&= 4
\end{aligned}
$$

If two letters is used to form a word, calculate ${}_4P_2$.

$$
\begin{aligned}
{}_4P_2 &= \frac{4!}{(4-2)!} \\
&= \frac{4!}{2!} \\
&= \frac{4 \times 3 \times 2!}{2!} \\
&= 4 \times 3 \\
&= 12
\end{aligned}
$$

If three letters is used to form a word, calculate ${}_4P_3$.

$$
\begin{aligned}
{}_4P_3 &= \frac{4!}{(4-3)!} \\
&= \frac{4!}{1!} \\
&= 4! \\
&= 4 \times 3 \times 2 \times 1 \\
&= 24
\end{aligned}
$$

If four letters is used to form a word, calculate ${}_4P_4$.

$$
\begin{aligned}
{}_4P_4 &= \frac{4!}{(4-4)!} \\
&= \frac{4!}{0!} \\
&= 4! \\
&= 24
\end{aligned}
$$

The total number of permutations is $4 + 12 + 24 + 24 = 64$.

4. How many different words can be formed using any number of letters in the word MANY? No letter is repeated.

Check your answer by turning to the Appendix.

Some questions involve permutations which affect each other, and the total number of permutations is the product of the permutations involved.

Look at the following example.

Example 4

How many different licence plates can be made using three letters from the word PAIR followed by three digits from the number 234?

Solution

$$
\begin{aligned}
\text{number of three-letter arrangements} &= {}_4P_3 \\
&= \frac{4!}{(4-3)!} \\
&= 24
\end{aligned}
$$

number of three-digit arrangements $= {}_3P_3$

$= 3!$

$= 6$

By the fundamental counting principle, the total number of arrangements is $24 \times 6 = 144$.

5. The licence plate of motorcycles in one province uses two letters followed by a two-digit number. How many different licence plates can be made using the letters M, T, and Z and the digits 3, 5, 7, and 9. (The letters and numbers do not repeat.)

Check your answer by turning to the Appendix.

When different groups of objects are arranged on a line and some groups are kept together, you can find the total permutation by following these steps:

Step 1: Find the permutation of each group.

Step 2: Decide on the number of positions a fixed group can be placed.

Step 3: Find the product of the results obtained in Steps 1 and 2.

Study the following example.

Example 5

In how many ways can a family of six stand in a row if the parents must always stand together?

Solution

The two parents and four children can stand in $2! \times 4!$ ways.

There are five positions among the four children to be filled.

Total Permutation $= 2! \times 4! \times 5$

$= 240$

The family can be arranged in 240 ways.

6. In how many ways can two boys and three girls be seated in a row if the three girls are always together?

7. Four different mathematics books and six different physics books are placed on a shelf such that the mathematics books are kept together on either end or in the middle of the set of physics books. How many such arrangements can be made?

Check your answer by turning to the Appendix.

Enrichment

In this section, you have studied a very important principle – the fundamental counting principle. It states that the number of ways of making two choices is $a \times b$ if the first choice can be made in a ways, and for each of these a ways, a second choice can be made in b ways. Since the result is the product of a and b, it is more appropriate to call it the multiplicative counting principle. There is another basic principle called the additive counting principle. Both principles can be explained in terms of sets.

The Multiplicative Counting Principle

If A and B are two sets of numbers such that $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$, then $n(A)$ is the number of elements in A and $n(B)$ is the number of elements in B. In this case, $n(A) = 3$ and $n(B) = 4$.

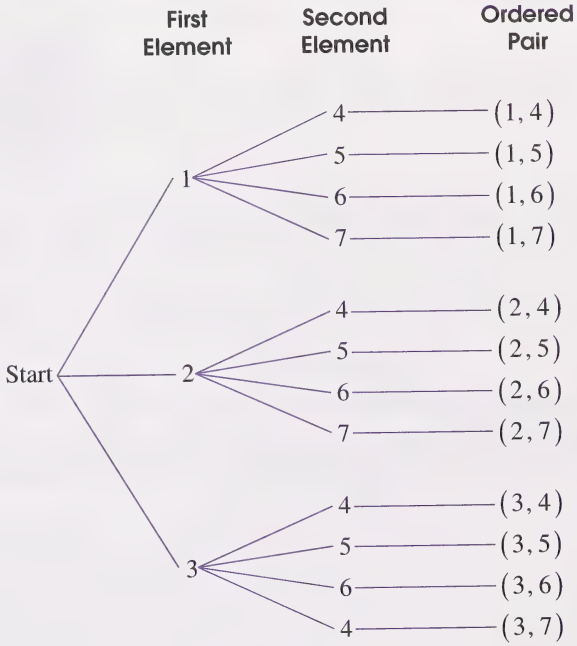
The Cartesian product $A \times B = \{(a, b) | a \in A,\ b \in B\}$ is the set of all the ordered pairs (a, b), and the number of elements in $A \times B$ is $n(A \times B) = n(A) \times n(B)$.

In this case, $n(A \times B) = 3 \times 4$

$= 12$

Therefore, there are twelve ordered pairs in the set of $A \times B$.

A tree diagram is used to show this Cartesian product.



The Additive Counting Principle

If A and B are two sets such that set A has $n(A)$ elements and set B has $n(B)$ elements, then

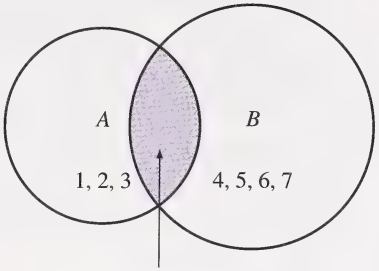
- $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$

$A \cup B$ = the union of A and B
$A \cap B$ = the intersection of A and B
ϕ = empty set, a set which has no element.

$A \cup B$ $A \cap B$

Consider the two sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$.

Use a diagram to illustrate this situation. Let circle A represent set A and circle B represent set B.



$(A \cap B) = \phi$

The intersection of A and $B = A \cap B = \phi$ because these two sets do not have any elements in common.

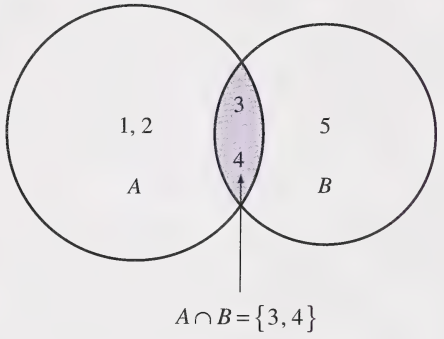
$n(A) = 3$ and $n(B) = 4$

$\therefore\ n(A \cup B) = 3 + 4$
$= 7$

The total number of different elements in the union of A and B is 7.

Now consider the two sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$. The intersection of A and $B = A \cap B = \{3, 4\}$ which is not an empty set.

The following diagram illustrates this situation.



$A \cap B = \{3, 4\}$

$n(A) = 4$, $n(B) = 3$, and $n(A \cap B) = 2$

$$\therefore\ n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 4 + 3 - 2$$
$$= 5$$

The total number of different elements in the union of A and B is 5.

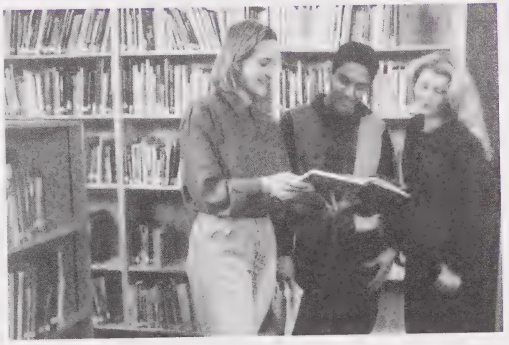
1. If $A = \{m, n, o\}$ and $B = \{h, f, g\}$, draw a tree diagram to illustrate the Cartesian product $A \times B$.

2. $M = \{2, 4, 6, 8, 10\}$, $P = \{1, 3, 5\}$, and $Q = \{1, 2, 3, 4\}$; find $n(P \cap Q)$, $n(M \cap P)$, $n(M \cup P)$, and $n(P \cup Q)$. Draw a diagram to illustrate each situation.

Check your answers by turning to the Appendix.

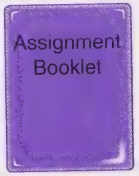
Conclusion

In this section you investigated permutations or arrangements. You applied the fundamental counting principle in problem situations, and used it to develop formulas for linear, circular, and ring permutations. You explored situations involving restrictions and like elements. You also applied the principles to answer probability questions.



The next time you are in a library, check to see if the books are shelved according to the Library of Congress system or the Dewey Decimal system. Can you imagine the chaos if the books were placed at random on the shelves? For even a medium-sized library, the number of arrangements would be astronomical! For instance, what is 10 000!?

Assignment



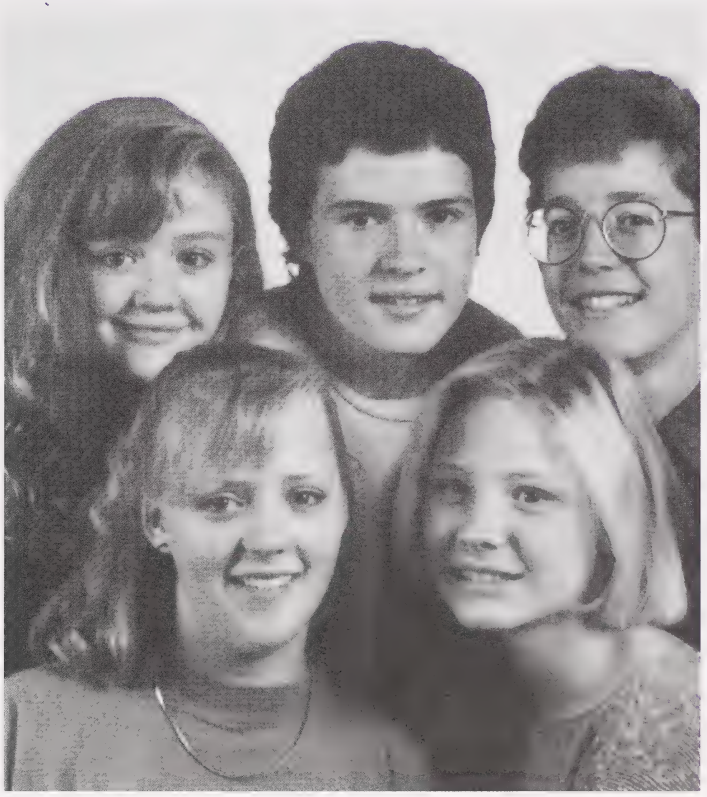
You are now ready to complete the section assignment.

Section 2: Combinations

A five-member student union executive is planning its Welcome Week activities. A subcommittee of three is to be struck to look after the decorations. In how many ways can this committee be chosen from the executive?

This question is an example of a selection or combination. Combination questions relate to the number of ways elements can be selected from a set. Selections are commonplace in everyday life: students are chosen for various assignments or committees; diners select items from a menu; pollsters select people to answer questionnaires.

In this section you will study combinations, and probability problems based on combinations. You will also study the Binomial Theorem. This theorem is based on combination theory and deals with finding powers of binomials.



WESTFILE INC.

Activity 1: Defining and Using the Combination Formula

If a student council wants to select a three-member committee to organize a graduation party, it is important to know who is selected, but the order of the selection is not relevant. Selections of this kind are called **combinations**.

Consider the four letters a, b, c, and d. In how many different ways can three letters be chosen from a, b, c, and d? The possible solutions are as follows:

$$(a,b,c)\ \ (b,c,d)\ \ (c,d,a)\ \ (d,a,b)$$

As you can see, there are four different ways to do it. If you try to arrange the letters in each group, each group can have 3! arrangements. The total number of arrangements should be $4 \text{ groups} \times 3! = 24$. This is the same as ${}_4P_3$. The four groups represent the number of combinations. If you use a new symbol ${}_nC_r$ to represent the number of combinations of n objects taken r at a time, then ${}_4C_3 = 4$.

The following is also true.

$$\begin{aligned} 4 \times 3! &= 24 \\ {}_4C_3 \times 3! &= 24 \\ &= {}_4P_3 \\ &= \frac{4!}{(4-3)!} \end{aligned}$$

$$\therefore\ {}_4C_3 = \frac{4!}{(4-3)!3!}$$

In general, for all natural numbers n and r, $n \geq r$.

$$\begin{aligned} {}_nC_r \times r! &= {}_nP_r \\ {}_nC_r \times r! &= \frac{n!}{(n-r)!} \\ {}_nC_r &= \frac{n!}{(n-r)!\,r!} \end{aligned}$$

${}_nC_r$ can also be written as $\binom{n}{r}$.

There is a problem. What is the number of combinations of n things taken zero at a time?

$$\begin{aligned} {}_nC_0 &= \frac{n!}{(n-0)!\,0!} \\ &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

Do you agree that there is only one way to choose nothing? Note that $0! = 1$.

The following two examples show you how to evaluate combinations using the formula ${}_nC_r = \frac{n!}{(n-r)!\,r!}$.

Example 1

Evaluate ${}_7C_5$.

Solution

$$
\begin{aligned}
{}_7C_5 &= \frac{7!}{(7-5)!5!} \\
&= \frac{7!}{2!5!} \\
&= \frac{7\times 6}{2!} \\
&= 21
\end{aligned}
$$

Example 2

Evaluate ${}_5C_1 + {}_3C_2 \times {}_4C_1 - {}_6C_0$.

Solution

$$
\begin{aligned}
&{}_5C_1 + {}_3C_2 \times {}_4C_1 - {}_6C_0 \\
&= \frac{5!}{(5-1)!1!} + \frac{3!}{(3-2)!2!} \times \frac{4!}{(4-1)!1!} - \frac{6!}{(6-0)!0!} \\
&= \frac{5}{1} + \frac{3}{1} \times \frac{4}{1} - 1 \\
&= 16
\end{aligned}
$$

1. Evaluate ${}_{17}C_3$.

2. Evaluate ${}_4C_3 + {}_5C_0 - {}_6C_1$.

3. Show that ${}_6C_3 + {}_6C_4 = {}_7C_4$.

Check your answers by turning to the Appendix.

When solving a combination problem, you only have to make a selection of objects from a given set of objects. You don't have to arrange these objects in definite order. Look at the following word problem.

Example 3

A committee of four students is to be chosen from a group of ten students. In how many ways can this be done?

Solution

There are ten students; thus, $n = 10$.

Four students are to be chosen; therefore, $r = 4$.

$$
\begin{aligned}
{}_{10}C_4 &= \frac{10!}{(10-4)!\,4!} \\
&= \frac{10!}{6!\,4!} \\
&= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} \\
&= 210
\end{aligned}
$$

There are 210 ways to form a committee.

4. A hockey league consists of nine teams. Each team plays exactly one game with each of the other teams. How many league games will be played?

5. A legislative body of twelve people passed a law by a vote of 8 to 4. In how many different ways could this vote have resulted?

6. In how many ways can a committee of six members be formed from a set of fifteen members?

Check your answers by turning to the Appendix.

The next example involves **mutually exclusive** cases. If there are two or more events, the occurrence of any one precludes the occurrence of any of the others. The word ***or*** will be used in the question. When you say that you want to select one kind **or** another, the selection of one kind of object excludes the selection of the other kind. The total number of selections is the sum of all the different selections.

Example 4

There are seven books on a bookshelf. In how many ways can five or more books be selected?

Solution

The statement **five or more** means you can select five books, six books, or seven books.

If you select five books, the number of selections is ${}_7C_5$.

If you select six books, the number of selections is ${}_7C_6$.

If you select seven books, the number of selections is ${}_7C_7$.

Altogether there are ${}_7C_5 + {}_7C_6 + {}_7C_7$ ways to choose the books.

$$
\begin{aligned}
{}_7C_5 + {}_7C_6 + {}_7C_7 &= \frac{7!}{2!\,5!} + \frac{7!}{1!\,6!} + \frac{7!}{0!\,7!} \\
&= \frac{7\times 6}{2\times 1} + 7 + 1 \\
&= 29
\end{aligned}
$$

The books may be chosen in 29 ways.

7. There are ten different pictures. In how many ways can seven or more pictures be selected?

8. In how many ways can Priscilla choose no more than three of twelve different cookies? (At least one cookie must be chosen.)

9. In how many different ways can three or more students be chosen out of eight students?

Check your answers by turning to the Appendix.

The next example is somewhat tricky.

Example 5

In how many ways can six different books be separated into two groups, each of three books?

Solution

Three books can be selected from the six books in ${}_6C_3$ ways. However, the books are separated into two groups of three books each. In other words, every time you select a group of three books from the six books, you actually get two groups of three books each. Therefore, the second group should not be counted again.

$$\therefore \text{ number of ways} = \frac{{}_6C_3}{2}$$

$$= \frac{\frac{6!}{3!3!}}{2}$$

$$= \frac{6\times5\times4}{3\times2\times2}$$

$$= 10$$

The books may be chosen in 10 ways.

Whenever a combination problem involves dividing a group of objects into two equal parts, you must divide ${}_nC_r$ by 2 in order to avoid counting identical cases twice.

10. In how many ways can ten basketball players be separated into two groups, each of five players?

11. In how many ways can a litter of eight puppies be separated into two groups, each of four puppies?

Check your answers by turning to the Appendix.

The next example is a problem involving multiple selections from different sources. The word ***and*** will be used in the question.

Example 6

Carolyn goes to a department store to buy two shirts, two pairs of jeans, and three pairs of socks. The store has eight shirts, ten pairs of jeans, and fifteen pairs of socks, all of different styles or colours. How many different selections of shirt, jeans, and socks can Carolyn purchase?

Solution

There are ${}_8C_2$ selections of shirts, ${}_{10}C_2$ selections of jeans, and ${}_{15}C_3$ selections of socks. For every selection of shirt, there are ${}_{10}C_2$ selections of jeans, and for every selection of shirt and jeans, there are ${}_{15}C_3$ selections of socks.

By the fundamental counting principle, you have to multiply all the selections in order to determine the total number of combinations.

$$\begin{aligned} {}_8C_2 \times {}_{10}C_2 \times {}_{15}C_3 &= \frac{8!}{(8-2)!2!} \times \frac{10!}{(10-2)!2!} \times \frac{15!}{(15-3)!3!} \\ &= \frac{8!}{6!2!} \times \frac{10!}{8!2!} \times \frac{15!}{12!3!} \\ &= 28 \times 45 \times 455 \\ &= 573\,300 \end{aligned}$$

Carolyn can purchase 573 300 combinations.

12. Krish goes to a grocery store to buy two kinds of cookies, one carton of milk, three kinds of vegetables, and four different kinds of soups. If the store has three different brands of milk, eight different kinds of cookies, ten different kinds of vegetables, and nine different kinds of soups, in how many different ways can Krish buy the groceries?

Check your answer by turning to the Appendix.

If a problem involves multiple selections from the same source, you also multiply the selections. Look at the following example.

Example 7

A bag of twenty different marbles is to be divided into a group of ten marbles, a group of six marbles, and a group of four marbles. In how many ways can this be done?

Solution

The first selection affects the second and third selections. The second selection also affects the third selection.

There are twenty different marbles. The first group can be selected in ${}_{20}C_{10}$ ways. Now, only ten marbles are left. Six marbles are chosen from the remaining ten marbles. The second group can be selected in ${}_{10}C_6$ ways. The third group can be chosen from the remaining four marbles in ${}_4C_4$ ways. Again, you multiply the selections.

$$\begin{aligned} {}_{20}C_{10} \times {}_{10}C_6 \times {}_4C_4 &= \frac{20!}{10!10!} \times \frac{10!}{6!4!} \times \frac{4!}{4!0!} \\ &= 38\,798\,760 \end{aligned}$$

The marbles can be divided in 38 798 760 ways.

13. A class of twenty-one students is to be divided into a group of nine students, a group of seven students, and a group of five students. In how many ways can this be done?

Check your answer by turning to the Appendix.

If any number of objects are selected from n objects, each object is either selected or excluded. For example, four objects a, b, c, and d are available. You want to know the number of ways to choose 1, 2, 3, 4, or no objects. Since each object can be either selected or excluded, then, by the fundamental counting principle, the total number of selections is the product of $2 \times 2 \times 2 \times 2 = 2^4$. This includes the case in which no object is chosen.

In general, if there are n objects, the total number of ways to select any number of objects is 2^n, which includes the case in which no object is chosen.

If you do not want the case in which no object is chosen to be included, then subtract 1 from 2^n. The number of ways to select any number of objects except zero is $2^n - 1$.

Example 8

In how many ways can a toddler choose one or more of five different toys?

Solution

Each toy can be dealt with in two ways. It can be chosen or rejected. The case in which no toy is chosen is not considered as a possible outcome.

$$\begin{aligned} \therefore \text{ number of ways} &= 2^5 - 1 \\ &= 32 - 1 \\ &= 31 \end{aligned}$$

The toddler can choose the toys in 31 ways.

14. In how many ways can Helena choose one or more of ten different candies?

15. In how many ways can Ralph choose one or more of seven electrical appliances?

16. A political party has eight candidates. In how many possible ways can 0, 1, 2,..., or 8 candidates be elected?

Check your answers by turning to the Appendix.

Some problems involve different kinds of restricted selections of objects. Study the next four examples carefully.

Example 9

A committee of four students is to be chosen from a group of twelve students including the president and secretary of the students' union.

- In how many ways can this be done if the president is to be included in every selection?
- In how many ways can this be done if the president and the secretary must be included?
- In how many ways can this be done if the president and the secretary must be excluded and one other student must be included?

Solution

In the first situation, the president is included in every selection; thus, three more students must be selected from the remaining eleven students.

$$\therefore \text{ number of selections} = {}_1C_1 \times {}_{11}C_3$$
$$= 1 \times \frac{11!}{8!3!}$$
$$= \frac{11 \times 10 \times 9}{3 \times 2 \times 1}$$
$$= 165$$

The selection can be made in 165 ways.

In the second situation, the president and the secretary must be included; thus, two more students must be selected from the remaining ten students.

$$\therefore \text{ number of selections} = {}_2C_2 \times {}_{10}C_2$$
$$= 1 \times \frac{10!}{8!2!}$$
$$= 45$$

The selection can be made in 45 ways.

In the third situation, one other student is always included and two are always excluded; thus, three more students must be chosen from the remaining nine students.

$$\therefore \text{ number of selections} = {}_1C_1 \times {}_9C_3$$
$$= 1 \times \frac{9!}{6!3!}$$
$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$
$$= 84$$

The selection can be made in 84 ways.

17. There are twelve face cards (four kings, four queens, and four jacks) in a standard deck of 52 cards. How many five-card hands can be formed under the following conditions?

a. No face cards are included.

b. Only two face cards are included.

c. The kings are always excluded and the two red queens are always included.

Check your answers by turning to the Appendix.

Some combination problems can be solved using the subtraction method instead of finding all of the possible combinations.

Study the following example.

Example 10

A class of twenty students—nine girls and eleven boys—is to select a committee of three students. At least one girl is to be on the committee. In how many ways can this be done?

Solution

Subtract the number of selections in which no girl is included from the number of selections of any three students with no restrictions.

The number of ways in which any three students can be selected out of twenty students is ${}_{20}C_3$.

The number of ways in which three students can be selected so that no girl is included is ${}_{20-9}C_3 = {}_{11}C_3$.

Find the number of ways in which three students can be selected so that at least one girl is included.

$$\begin{aligned} {}_{20}C_3 - {}_{11}C_3 &= \frac{20!}{17!3!} - \frac{11!}{8!3!} \\ &= \frac{20\times19\times18}{3\times2\times1} - \frac{11\times10\times9}{3\times2\times1} \\ &= 1140 - 165 \\ &= 975 \end{aligned}$$

The committee can be formed in 975 ways.

Alternate Method

$${}_9C_1 \times {}_{11}C_2 + {}_9C_2 \times {}_{11}C_1 + {}_9C_3$$

↑ one girl ↑ two girls ↑ three girls

$$= 495 + 396 + 84$$
$$= 975$$

18. How many five-card hands that contain at least one king can be formed from a standard deck?

19. A school cafeteria offers five different drinks, three kinds of sandwiches, and six kinds of desserts. A lunch package contains five items. How many different packages can be formed if one drink and at least one sandwich must be included? (No repetition is allowed.)

20. Tui has five shirts, six pairs of pants, and three hats. How many selections of shirts, pants, and hats can be made, taking at least one shirt and one pair of pants?

Check your answers by turning to the Appendix.

The next two examples deal with another special kind of problem involving geometric figures.

Example 11

There are eight coplanar points. Three of them are collinear. No three of the other five points are in a straight line. How many straight lines can be formed by joining any two of the eight points?

Solution

Method 1

If no three of the eight points were collinear, the number of lines formed would be ${}_8C_2$.

The number of lines formed by three noncollinear points is ${}_3C_2$.

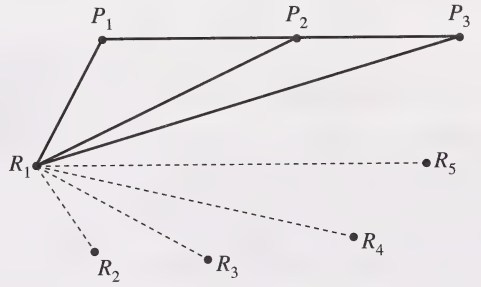
Since the three points are collinear, they form one straight line.

$$\begin{aligned}\therefore \text{ number of lines} &= {}_8C_2 - {}_3C_2 + 1 \\ &= \frac{8!}{6!2!} - \frac{3!}{1!2!} + 1 \\ &= 28 - 3 + 1 \\ &= 26\end{aligned}$$

A total of 26 straight lines are formed by joining the eight points if three of them are collinear.

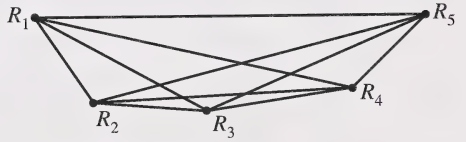
Method 2

In the following diagram, P_1, P_2, and P_3 represent the three collinear points. The noncollinear points are represented by R_1, R_2, R_3, R_4, and R_5.



If R_1 is joined to each collinear point (P_1, P_2, P_3), three lines are formed. When each of the five noncollinear points $(R_1, R_2, R_3, R_4, R_5)$ are joined to each of the three collinear points (P_1, P_2, P_3), the number of lines formed is $5 \times 3 = 15$.

One line is produced by joining any two of the noncollinear points $(R_1, R_2, R_3, R_4, R_5)$. The total number of lines produced by the five noncollinear points is ${}_5C_2 = \frac{5!}{3!2!} = 10$.



The three collinear points P_1, P_2, and P_3 form one straight line.

$$\therefore \text{ number of lines} = 15 + 10 + 1$$
$$= 26$$

The number of lines formed is 26.

Example 12

Determine the number of diagonals in a nonagon (nine sides).

Solution

A nonagon has nine vertices. The number of lines formed by joining any two of these vertices including the sides is ${}_9C_2$.

$$\therefore \text{ number of diagonals} = \text{number of lines} - \text{number of sides}$$
$$= {}_9C_2 - 9$$
$$= \frac{9!}{7!2!} - 9$$
$$= 36 - 9$$
$$= 27$$

There are 27 diagonals in a nonagon.

21. There are twelve coplanar points. Four of them are collinear. No three of the other eight points are in a straight line. How many straight lines can be formed by joining any two of the twelve points?

22. Determine the number of diagonals in a fourteen-sided polygon.

23. How many parallelograms can be formed by using two sets of parallel lines, if one set has five parallel lines and another set has six parallel lines?

Check your answers by turning to the Appendix.

You were given a variety of problems in this activity. You should have the necessary skills to solve similar combination problems.

Activity 2: Probability Using Combinations

In probability, some sample spaces involve a large number of elements. It is impractical to list all the elements of a large sample space. Permutations and combinations can be used to find the number of favourable or possible cases. Look at the following examples.

Example 1

Three cards are drawn at random from a well-shuffled ordinary deck of 52 cards. Find the probability that all three cards are kings.

Solution

Method 1: Using Combinations

Find the number of ways in which three cards can be selected out of 52 cards.

$$
\begin{aligned}
{}_{52}C_3 &= \frac{52!}{49!3!} \\
&= 22\,100 \\
&= n(S)
\end{aligned}
$$

Find the number of ways in which three kings can be selected out of four kings.

$$
\begin{aligned}
{}_4C_3 &= \frac{4!}{3!1!} \\
&= 4 \\
&= n(A)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ required probability } P(E) &= \frac{n(A)}{n(S)} \\
&= \frac{4}{22\,100} \\
&= \frac{1}{5525}
\end{aligned}
$$

The probability that all three cards are kings is $\frac{1}{5525}$.

Method 2: Using Permutations

Find the number of arrangements of 52 cards, taken three at a time.

$$
\begin{aligned}
{}_{52}P_3 &= \frac{52!}{(52-3)!} \\
&= 132\,600
\end{aligned}
$$

Find the number of arrangements of four kings, taken three at a time.

$$
\begin{aligned}
{}_4P_3 &= \frac{4!}{1!} \\
&= 24
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ required probability } P(E) &= \frac{24}{132\,600} \\
&= \frac{1}{5525}
\end{aligned}
$$

The probability that all three cards are kings is $\frac{1}{5525}$.

Example 2

There are twelve boys and eight girls in a class. If a group of five students is chosen at random to play a game, find the probability that two of them are girls.

Solution

Find the number of ways in which five students can be chosen out of twenty.

$$
\begin{aligned}
{}_{20}C_5 &= \frac{20\times19\times18\times17\times16}{5\times4\times3\times2\times1} \\
&= 15\,504
\end{aligned}
$$

Find the number of ways in which two girls and three boys can be chosen.

$$
\begin{aligned}
{}_{12}C_3 \times {}_8C_2 &= \frac{12!}{9!3!}\times\frac{8!}{6!2!} \\
&= 220\times28 \\
&= 6160
\end{aligned}
$$

$$
\begin{aligned}
\therefore \text{ required probability } P(E) &= \frac{6160}{15\,504} \\
&= \frac{385}{969}
\end{aligned}
$$

The probability that two girls are chosen is $\frac{385}{969}$.

1. A bag contains nine cards, numbered from 1 to 9 inclusive. Two cards are drawn at random. What is the probability that both cards are even numbers?

2. Jody is one of twenty-five students in a class. Five students will be chosen at random to form a basketball team. What is the probability that Jody will be a member of the team?

3. Horst bought two of a total of fifty raffle tickets. If there are two winning tickets, what is the probability that Horst has exactly one winning ticket?

4. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability that both cards are red.

Check your answers by turning to the Appendix.

Some questions involve more than one event. When the word ***or*** is used in the question, you have to apply the addition method.

Example 3

If two cards are drawn at random from an ordinary deck of 52 cards, what is the probability that both are kings or both are jacks?

Solution

Find the number of combinations of 52 cards taken two at a time.

$$_{52}C_2 = \frac{52!}{50!2!}$$
$$= 1326$$

Find the number of combinations of four kings taken two at a time.

$$_4C_2 = \frac{4!}{2!2!}$$
$$= 6$$

Find the number of combinations of four jacks taken two at a time.

$$_4C_2 = \frac{4!}{2!2!}$$
$$= 6$$

$$\therefore P(E) = \frac{6}{1326} + \frac{6}{1326}$$
$$= \frac{12}{1326}$$
$$= \frac{2}{221}$$

The probability that both cards are kings or jacks is $\frac{3}{221}$.

5. A marble is drawn at random from a bag which contains four red marbles, six white marbles, and two green marbles. What is the probability that the marble drawn is red or green?

Check your answer by turning to the Appendix.

You apply the multiplication method when the word ***and*** is used. This is shown in the following example.

Example 4

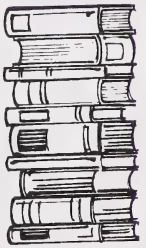
What is the probability of selecting two math books and four physics books from a set of five different math books and five different physics books?

Solution

Any six books can be selected in $_{10}C_6$ ways.

The two math books can be selected in $_5C_2$ ways.

The four physics books can be selected in $_5C_4$ ways.



$$\therefore P(E) = \frac{{}_5C_2 \times {}_5C_4}{{}_{10}C_6}$$

$$= \frac{\frac{5!}{3!2!} \times \frac{5!}{1!4!}}{\frac{10!}{4!6!}}$$

$$= \frac{10 \times 5}{210}$$

$$= \frac{5}{21}$$

The probability of the selection is $\frac{5}{21}$.

6. A committee of four members is to be chosen from four Alberta delegates, three N.W.T. delegates, and three Newfoundland delegates. What is the probability of choosing two delegates from Alberta, one from N.W.T., and one from Newfoundland?

7. A coin is tossed; then, a die is rolled. What is the probability that a tail and an odd number turn up?

Check your answers by turning to the Appendix.

If a sample space is separated into two parts, E and $\overline{E}$, then E is the complement of $\overline{E}$, and vice versa. The probability of a complementary event $P(\overline{E})$ can be determined by subtracting the probability of event E from 1: $P(\overline{E}) = 1 - P(E)$.

The following example involves complementary events.

Example 5

Three toy cars are chosen at random from nine toy cars of which four are defective. Find the probability that at least one is defective.

Solution

In order to find the probability that at least one toy car is defective, you have to find the probability that no toy car is defective. Subtract this probability from 1. This will give the probability that at least one toy car is defective.

If four toy cars are defective, then five cars are not defective.

Find the number of ways to choose three nondefective toy cars.

$${}_5C_3 = \frac{5!}{2!3!}$$

$$= 10$$

Find the number of ways to choose any three toy cars.

$${}_9C_3 = \frac{9!}{6!3!}$$

$$= 84$$

The probability that no toy car is defective is $\frac{10}{84}$.

Find the probability that at least one toy car is defective.

$$1-\frac{10}{84}=\frac{74}{84}$$
$$=\frac{37}{42}$$

The probability that at least one car is defective is $\frac{37}{42}$.

8. There are fifteen boys and ten girls in a class. A yearbook committee of four is chosen at random. What is the probability that at least one girl is chosen?

Check your answer by turning to the Appendix.

In this activity you discovered that combinations can also be used to find the probability of an event.

Activity 3: Binomial Theorem and Pascal's Triangle

Binomial Theorem

A binomial is a polynomial which has two terms. In this section, you are going to investigate the expansion of binomials of the form $(a+b)^2$, $(a+b)^3$, and so on. Consider the following expansions:

$$(a+b)^2=(a+b)(a+b)$$
$$=a^2+ab+ba+b^2$$
$$=a^2+2ab+b^2$$

Each term of the expansion is obtained by multiplying a letter (a or b) from each of the two factors $(a+b)$. The first term of the product $\left(a^2\right)$ has no bs. The second term $(2ab)$ has one b, and the third term $\left(b^2\right)$ has two bs.

$$(a+b)^3=(a+b)(a+b)(a+b)$$
$$=aaa+aab+aba+abb+baa+bab+bba+bbb$$
$$=a^3+3a^2b+3ab^2+b^3$$

Again, each term of the expansion is obtained by multiplying a letter (a or b) from each of the three factors $(a+b)$. The first term of the product $\left(a^3\right)$ has no bs. The second term $\left(3a^2b\right)$ has one b. The third term $\left(3ab^2\right)$ has two bs. The fourth term $\left(b^3\right)$ has three bs.

Each term looks like a combination of as and bs as if all the as and bs were different.

The first term has no bs. It is like choosing no bs from three bs. The combination ${}_3C_0 = 1$ is the coefficient of the first term a^3.

The second term has one b. It is like choosing one b from three bs. The combination ${}_3C_1 = 3$ is the coefficient of the second term $3a^2b$.

The third term has two bs. It is like choosing two bs from three bs. The combination ${}_3C_2 = 3$ is the coefficient of the third term $3ab^2$.

The fourth term has three bs. It is like choosing three bs from three bs. The combination ${}_3C_3 = 1$ is the coefficient of the fourth term b^3.

If you use the same reasoning to expand $(a+b)^4$, then

$$\begin{aligned}(a+b)^4 &= {}_4C_0a^4 + {}_4C_1a^3b + {}_4C_2a^2b^2 + {}_4C_3ab^3 + {}_4C_4b^4\\ &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

(If you wish, multiply $(a+b)(a+b)(a+b)(a+b)$. The answer should be the same.)

Using the same reasoning, you can expand $(a+b)^n$.

The first term has no bs. The coefficient is ${}_nC_0 = 1$ and a^n will be the first term.

The second term has one b. The coefficient is ${}_nC_1 = n$ and $na^{n-1}b$ will be the second term.

The third term has two bs. The coefficient is ${}_nC_2$ and ${}_nC_2a^{n-2}b^2$ will be the third term.

Continued this way, the rth term has $(r-1)$ bs.

The last term has n bs. The coefficient of the last term is ${}_nC_n$ and ${}_nC_nb^n$ is the nth term.

In general,

$$\begin{aligned}(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots\\ + {}_nC_ka^{n-k}b^k + \ldots + {}_nC_nb^n,\end{aligned}$$

where $n \in I$ and $n \geq 0$. This expansion is called the **Binomial Theorem**.

Look at the following examples.

Example 1

Use the Binomial Theorem to expand $(a+b)^6$.

Solution

$$(a+b)^6$$
$$= {}_6C_0 a^6 + {}_6C_1 a^5 b + {}_6C_2 a^4 b^2 + {}_6C_3 a^3 b^3 + {}_6C_4 a^2 b^4 + {}_6C_5 a b^5 + {}_6C_6 b^6$$
$$= a^6 + 6a^5 b + 15a^4 b^2 + 20a^3 b^3 + 15a^2 b^4 + 6a^1 b^5 + b^6$$

Example 2

Determine the first three terms in the expansion of $(3x-5)^{10}$ by using the Binomial Theorem.

Solution

$$(3x-5)^{10}$$
$$= \left[(3x)+(-5)\right]^{10}$$
$$= {}_{10}C_0 (3x)^{10} + {}_{10}C_1 (3x)^9 (-5) + {}_{10}C_2 (3x)^8 (-5)^2 + \ldots$$
$$= 59\,049x^{10} + (10)(19\,683)x^9 (-5) + (45)(6561)x^8 (25) + \ldots$$
$$= 59\,049x^{10} - 984\,150x^9 + 7\,381\,125x^8 \ldots$$

1. Use the Binomial Theorem to expand each of the following:

   a. $(x+2y)^8$

   b. $(2x-y)^7$

   c. $\left(x^2 - \frac{1}{2}y\right)^5$

2. Use the Binomial Theorem to expand $(a+b-c)^4$.

3. Use the binomial expansion of $(x+y)^2$ to prove that
   $2^7 = {}_7C_0 + {}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4 + {}_7C_5 + {}_7C_6 + {}_7C_7$.

Check your answers by turning to the Appendix.

The **general term** of the binomial expansion is ${}_nC_k a^{n-k} b^k$. By letting $k = 0, 1, 2, \ldots, n$, you can construct any term of the series. You must note that the formula, ${}_nC_k a^{n-k} b^k$ determines the $(k+1)$th term in the expansion. For example, in the Binomial Theorem, ${}_nC_3$ is the coefficient of the $(3+1)$ or fourth term.

The next example determines the term of a binomial expansion.

Example 3

Determine the seventh term (t_7) in the expansion of $(2x-3)^{11}$.

Solution

The seventh term of the expansion is generated by substituting $k=6$ in the general term formula.

Substitute $n=11$, $a=2x$, $b=-3$, and $k=6$ in ${}_nC_k a^{n-k} b^k$.

$$\begin{aligned} t_7 &= {}_{11}C_6 (2x)^{11-6} (-3)^6 \\ &= \frac{11!}{5!6!}(2x)^5 (-3)^6 \\ &= 462\left(32x^5\right)(729) \\ &= 10\,777\,536x^5 \end{aligned}$$

Note: In the formula sheet of the Mathematics 30 Diploma Exam, the general term in the expansion of $(x+y)^n$ is $t_{k+1} = {}_nC_k x^{n-k} y^k$.

If you use the t_{k+1}, then the seventh term is as follows:

$$\begin{aligned} t_7 &= t_{6+1} \\ &= {}_{11}C_6 (2x)^{11-6} (-3)^6 \\ &= 10\,777\,536x^5 \end{aligned}$$

4. Determine the ninth term in the expansion of $(x-2)^{15}$.

5. Determine the middle term of $\left(x^2 - \sqrt{y}\right)^6$.

Check your answers by turning to the Appendix.

In the expansion of a binomial, the coefficient is determined by the formula ${}_nC_k$.

Study the next two examples.

Example 4

What is the numerical coefficient, for the term x^5y^7 in the expression of $(x+y)^{12}$?

Solution

Since the exponent of y in x^5y^7 is 7, then $k=7$.

The $(k+1)$th term is $7+1=8$.

Thus, the coefficient of the eighth term is ${}_{12}C_7$.

$$\begin{aligned} {}_{12}C_7 &= \frac{12!}{(12-7)!7!} \\ &= \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1} \\ &= 792 \end{aligned}$$

Example 5

${}_{15}C_k x^6 y^k$ is a term in the expansion of $\left(x^2 + y\right)^{15}$. What is the value of k?

Solution

In the expansion of $(x+y)^n$, the sum of the exponents for any one term in the expansion is always equal to the value of n.

Since x^2 is the first term, ${}_{15}C_k x^6 y^k$ must be rewritten as ${}_{15}C_k \left(x^2\right)^3 y^k$.

$$k+3=15$$
$$k=12$$

6. What is the numerical coefficient, in the form of ${}_nC_k$, for the term $a^8 b^8$ in the expansion of $(a-b)^{16}$?

7. If the numerical coefficient is not stated, $a^x b^7$ is a term in the expansion of $(a+b)^{12}$. What is the value of x?

8. Which term in the expansion of $\left(\frac{1}{2}x - y\right)^6$ is $\frac{15}{16}x^4 y^2$?

9. To what other combination is ${}_{14}C_6$ equivalent?

Check your answers by turning to the Appendix.

Pascal's Triangle

When planning a vacation where you are driving to your destination, you may choose between two different routes: a scenic route or a shortcut. When expanding binomials, you may wish to include all the detail by using the Binomial Theorem—the scenic route. An alternative method, or shortcut, is to use Pascal's Triangle—a quick method of evaluating the coefficients of an expression.

The numerical coefficients in the binomial expansion $(a+b)^n$ can be written in the form of a triangular array. This array is called **Pascal's Triangle**. The Chinese developed a similar array in the tenth century; it was mentioned in a twelfth century Chinese manuscript. However, the credit for developing this triangle and further applications of the triangle is given to Blaise Pascal (1623 to 1662 A.D.).

If you arrange the numerical coefficients in the binomial expansion $(a+b)^n$ in a triangular array, you can see some interesting patterns.

$(a+b)^0$ $\quad n=0$

$(a+b)^1$ $\quad n=1$

$(a+b)^2$ $\quad n=2$

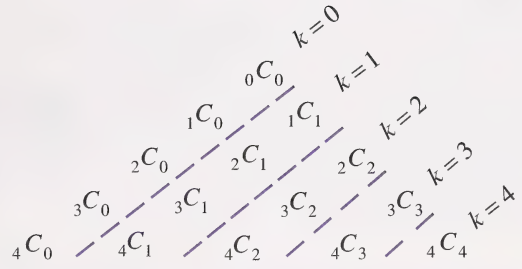
$(a+b)^3$ $\quad n=3$

$(a+b)^4$ $\quad n=4$

. .

. .

. .



If you evalute each ${}_nC_k$, the triangle becomes as follows:

$n=0$ $\qquad$ 1

$n=1$ $\qquad$ 1 $\quad$ 1

$n=2$ $\qquad$ 1 $\quad$ 2 $\quad$ 1

$n=3$ $\qquad$ 1 $\quad$ 3 $\quad$ 3 $\quad$ 1

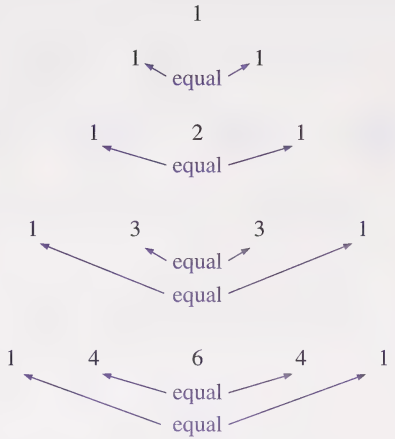
$n=4$ $\qquad$ 1 $\quad$ 4 $\quad$ 6 $\quad$ 4 $\quad$ 1

. .

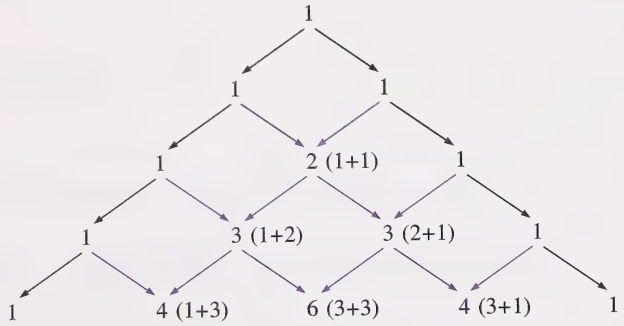
. .

. .

In any row of this triangle, the coefficients are the same read forwards or backwards.



You can extend the triangle by adding two adjacent numbers to obtain the number in the row below. This is shown in the following diagram.



You can use Pascal's Triangle and the pattern of exponents to expand binomials.

Study the following examples.

Example 6

Use Pascal's Triangle to expand $(x+y)^3$.

Solution

You know that the sum of the exponents of m and n in each term is 3.

$$(x+y)^3 = \square\, x^3 + \square\, x^2y + \square\, xy^2 + \square\, y^3$$

The coefficients for each term are obtained from row 4 ($n=3$) of Pascal's Triangle (as shown in the preceding diagram).

The expansion of $(x+y)^3$ is $x^3+3x^2y+3xy^2+y^3$.

Example 7

Use Pascal's Triangle to expand $(u+v)^5$.

Solution

The sum of the exponents for u and v in each term is 5.

$$(u+v)^5 = \square\, u^5 + \square\, u^4v + \square\, u^3v^2 + \square\, u^2v^3 + \square\, uv^4 + \square\, v^5$$

Use Pascal's Triangle to determine the unknown coefficients. Use the row $n=4$ to get the row for $n=5$.

$n=4$: 1 4 6 4 1

$n=5$: 1 5 10 10 5 1

$$(u+v)^5 = u^5+5u^4v+10u^3v^2+10u^2v^3+5uv^4+v^5$$

10. Determine the missing exponent x for the given terms of the expansion $(r+s)^{15}$.

a. $r^{10}s^{x}$ **b.** $r^{x}s^{8}$

11. Use Pascal's Triangle to expand $(x+y)^{7}$.

Check your answers by turning to the Appendix.

Did you find the patterns in Pascal's Triangle interesting? You can expand any binomial of the form $(x+a)^{n}$ using the Binomial Theorem.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may find it helpful to view the video titled *Permutations and Combinations*, from the *Catch 30* series, ACCESS Network. This program will help reinforce the concepts of permutations and combinations. This video is available from the Learning Resources Distributing Centre.

When r objects are selected from n objects $(n \geq r)$ and the order of the objects is not important, this selection is called a combination. The symbol used to denote the number of combinations of n objects taken r at a time is ${}_nC_r$, and ${}_nC_r = \frac{n!}{(n-r)!r!}$.

If a question involves more than one kind of selection and no two kinds of selections can occur at the same time, the word ***or*** will be used in the question, and the total number of selections is the sum of the numbers of different selections.

Look at the following example.

Example 1

There are twenty-four colouring pencils in a box. In how many ways can twenty or more pencils be selected?

Solution

You may select 20, 21, 22, 23, or 24 pencils.

$\therefore$ number of selections

$$= {}_{24}C_{20} + {}_{24}C_{21} + {}_{24}C_{22} + {}_{24}C_{23} + {}_{24}C_{24}$$

$$= \frac{24!}{4!\,20!} + \frac{24!}{3!\,21!} + \frac{24!}{2!\,22!} + \frac{24!}{1!\,23!} + \frac{24!}{0!\,24!}$$

$$= 12\,951$$

The pencils may be selected in 12 951 ways.

1. A student wants to choose three or more programs out of five programs available in school. In how many ways can this be done?

Check your answer by turning to the Appendix.

When different selections are made from different sources, the number of combinations is the product of the combinations. The word ***and*** will usually be used in the question.

The next example shows these kinds of selections.

Example 2

Mike has six different colouring pencils and four sheets of different coloured paper. He needs one colouring pencil and one sheet of paper. In how many ways can he select the pencil and paper?

Solution

There are ${}_6C_1$ ways to select a colouring pencil.

There are ${}_4C_1$ ways to select a sheet of paper.

$$\therefore \text{ number of selections} = {}_6C_1 \times {}_4C_1$$
$$= \frac{6!}{5!1!} \times \frac{4!}{3!1!}$$
$$= 24$$

Mike can select the pencil and paper in 24 ways.

2. In how many ways can two shirts and three ties be selected from five shirts and six ties?

Check your answer by turning to the Appendix.

The total number of ways to select any number of n objects is 2^n, which includes the case in which no object is selected. If you want the number of ways to select one or more of n objects, the total number is $2^n - 1$.

Study the following example

Example 3

In how many ways can a student choose one or more of six courses?

Solution

Each course can be chosen or rejected.

The student wants to choose at least one course.

The total number of ways is $2^6 - 1 = 63$ ways.

3. In how many ways can you choose one or more pens from four different pens?

Check your answer by turning to the Appendix.

When the words ***at least*** are used in the question, usually it means that many cases are involved. It may be easier to use the subtraction method, as shown in Example 4.

Example 4

There are ten math books and eight chemistry books. In how many ways can five books be chosen with at least one math book?

Solution

There are eighteen books altogether.

The number of ways to choose any five books is ${}_{18}C_5$.

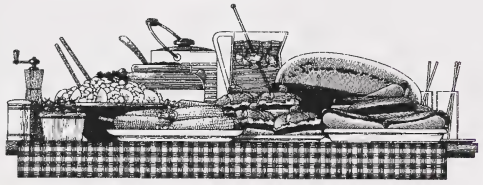
The number of ways to choose no math books equals the number of ways to choose five chemistry books, which is ${}_{8}C_5$.

Find the number of ways to choose five books with at least one math book.

$$\begin{aligned} {}_{18}C_5 - {}_{8}C_5 &= 8568 - 56 \\ &= 8512 \end{aligned}$$

The choice can be made in 8512 ways.

4. There are eleven main dishes and five beverages. In how many ways can five items be chosen with at least one beverage?



Check your answer by turning to the Appendix.

Enrichment

Some problems involve both permutations and combinations. These questions are usually more challenging. To solve this type of question, usually you have to calculate the number of combinations first.

Look at the following examples.

Example 1

Given seven consonants and four vowels, how many five-letter arrangements can be made if each arrangement consists of three different consonants and two different vowels?

Solution

The three consonants can be selected in ${}_7C_3$ ways.

The two vowels can be selected in ${}_4C_2$ ways.

The five letters can be arranged in ${}_5P_5$ ways.

$$\begin{aligned} \therefore \text{ number of arrangements} &= {}_7C_3 \times {}_4C_2 \times {}_5P_5 \\ &= 35 \times 6 \times 120 \\ &= 25\,200 \end{aligned}$$

A total of 25 200 arrangements can be made.

Example 2

A round table can seat seven, and ten people are available. In how many ways can a seating arrangement be made?

Solution

The seven people can be selected in ${}_{10}C_7$ ways.

The round table can have $(7-1)!$ arrangements.

$$\begin{aligned} \therefore \text{ number of arrangements} &= {}_{10}C_7 \times (7-1)! \\ &= 120 \times 720 \\ &= 86\,400 \end{aligned}$$

Seating arrangements can be made in 86 400 ways.

1. There are twenty-six students in a class. Six students will be selected to form a committee. Two members of the committee will be designated as chairperson and as secretary. In how many ways can this be done?

2. Linda wants to string ten different charms to make a charm bracelet. If there are fifteen different charms available, how many different arrangements can be made?

3. In how many ways can four books be selected and arranged on a bookshelf if there are three different novels and two different physics books, and it is required that the two physics books are selected and placed together on the shelf?

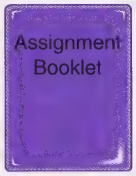
Check your answers by turning to the Appendix.

Conclusion

In this section you developed and applied the formula for the number of combinations of a set of objects. Some of the combination problems that you solved were complicated by restrictions imposed on the way the items were to be chosen. You applied the principles of combinations in probability questions and when you developed the Binomial Theorem and Pascal's Triangle.

You have discovered that a combination is a selection or choice. For instance, a class may have thirty students. When a teacher selects two students to perform an experiment in front of the class, the "volunteers" can truly view themselves as part of a very select group! You see, the teacher had ${}_{30}C_2$ or 435 choices!

Assignment



You are now ready to complete the section assignment.

Section 3: Statistics

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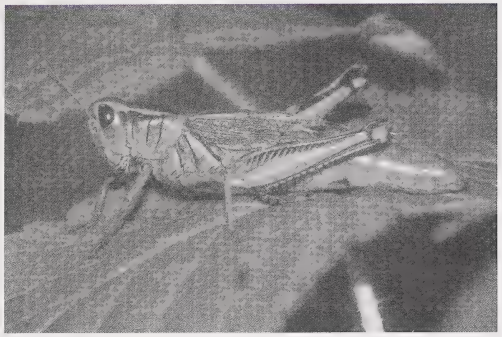
A large appliance company determines that, in order to improve its sales of an electric range it manufactures, the company must extend its warrantee. What factors must the manufacturer take into account, particularly if the decision it makes must not adversely affect the profits? The company does not want to repair or replace too many of its ranges. Provided the manufacturer has good statistics on the durability of its range, and on how long the average range works before it needs to be repaired, the company can make an appropriate decision.

In today's world, collection and interpretation of statistics are becoming increasingly important. People in all fields—business, the sciences and research, sports and entertainment, to name but a few—use statistical information daily. Statistics is the branch of mathematics dealing with the collection, analysis, interpretation, and presentation of data.

In this section you will investigate the branch of statistics that deals with how the data collected from a population is spread across that population. For example, information about heights and weights of people deals with averages and the proportions of the population that fall into various categories. A typical distribution, called the standard normal distribution, and two measures—the mean and standard deviation—are the focus of this section.

Activity 1: Mean and Standard Deviation



Plants depend on the weather for survival. Under ideal conditions, plants flourish; and so, too, do some insects. Most insects are dependent on plants for food. The mean population of insects varies from year to year depending on the amount of food available to them.

The **mean** of a set of scores is the sum of the scores divided by the number of scores. The sumbol μ, which is read **mu**, is used to represent the mean of a sample.

Standard deviation is a measure that describes the distribution of data about the mean. It determines how many units each observation in a sample deviates from the mean.

The symbol σ (sigma) denotes standard deviation; σ is the lower case of $\sum$ (sum of) in the Greek alphabet. The greater the standard deviation, the greater the dispersion or spread of scores; thus, the results will vary. A smaller standard deviation is closer to the mean; the result is more reliable and is more consistent.

Many calculators now contain statistical functions, but you must be careful when you are using some calculator since you may not be able to see the information that you have entered. Some calculator may be designed to give you only the answers to the different types of functions.

The procedure for only one type of calculator, the CASIO *fx-7700G* graphing calculator, is shown in this activity. If you have a different calculator, check the owner's manual to see the correct method.

Note: If you choose to use a scientific calculator, then refer to the owner's manual for the proper procedure.

Example

Use a graphing calculator to answer this question.

A small group of investors is planning to build a ski resort. The group has found only three locations that meet its criteria. The decision on which area to purchase is going to be based on the amount of snowfall on the slopes. The amount of snowfall (in centimetres) on each of the slopes over the last fifteen years is given.

Glacier Site

292, 255, 307, 318, 353,
281, 342, 329, 268, 289,
321, 312, 298, 319, 291

Sunridge Site

305, 237, 348, 373, 279,
262, 331, 339, 271, 357,
253, 219, 391, 336, 274

Alpine Basin Site

321, 267, 232, 304, 289,
342, 380, 403, 207, 294,
316, 237, 373, 251, 359

Based on this data, which site will the group select?

Solution

Use a graphing calculator to calculate the standard deviation and mean for each site.

The procedure to find the standard deviation and mean using the CASIO *fx-7700G* graphing calculator is as follows:

Step 1: Turn on the calculator.

Step 2: Press Mode ×.

Run/SD will appear in the first row of the display to inform you that the calculator is in its statistical mode.

Step 3: Press Mode Shift 2.

This informs the calculator that the statistical data is input; you do not want to store the data.

Step 4: Press Shift 3 F2.

Scl will appear on the upper left corner of the display. Scl means the statistical memory is cleared.

Step 5: Press EXE PRE.

The number **0.** will be displayed.

Step 6: Enter the values, followed by F1 after each piece of data.

Enter the values for Glacier Site.

2 9 2 F1 2 5 5 F1 ...
... 3 1 9 F1 2 9 1 F1

Step 7: Press F4 F2 EXE.

The number **25.868 900 25** is displayed. This is the standard deviation for the set of data. The calculator shows the symbol $\times\sigma n$ for standard deviation.

Step 8: Press F1 EXE.

The number **305.** is displayed. This is the mean $\overline{X}$ for the set of data.

Step 9: Repeat the steps for Sunridge Site and Alpine Basin Site.

Using these steps, the standard deviations for Sunridge Site and Alpine Basin Site are approximately 50.8 and 57.0. The mean for each of these sites is 305.

Since Glacier Site has the lowest standard deviation, its snowfall is more consistent than the other two sites. A lower standard deviation gives a more reliable result because it is closer to the mean. The snowfall in Galcier Site does not vary as much as the others.

1. A golfer got the following scores in the last twelve rounds.

102	96	105	105	96	111
106	100	99	103	102	98

Calculate the standard deviation to one decimal place.

2. The masses (in grams) of potato chips packaged by two different companies are given in the chart. Which company is more consistent in its packaging process?

Crunchies				Munchies			
198	205	201	200	199	201	201	197
207	192	194	199	201	198	196	203
206	210	198	193	198	206	203	196
195	192	207	203	206	197	202	196

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Check your answers by turning to the Appendix.

This activity introduced the standard deviation symbol σ, and provided some opportunities for finding σ using a calculator.

Activity 2: Identifying Normal Distributions

The number of data involved in the previous activities was kept to a minimum. What happens when you have to deal with thousands or millions of data?

A new method has to be developed to deal with these situations.

Take a look at the following situation.

The data collected represents the number of skis sold by a small ski shop.

Ski Length (cm)	Number Sold
140	1
150	8
160	28
170	56
180	70
190	56
200	28
210	8
220	1

What percentage of the frequency falls within one standard deviation of the mean?

The standard deviation σ for the set of data is 14.1.

The mean length μ of the skis sold is 180 cm. You are interested in the number that fall between 165.9 cm (180 cm − 14.1 cm) and 194.1 cm (180 cm + 14.1 cm).

How many skis fall into this range?

Ski Length (cm)	Number Sold
170	56
180	70
190	56
Total	182

Therefore, 182 skis fall into this range.

What percentage is this of the total number of skis sold?

The shop sold a total of 256 skis.

$$\text{Percentage} = \frac{182}{256} \times 100\%$$
$$\doteq 71\%$$

If you made a frequency polygon (ski length versus frequency) of the set of data, it would have a bell shape or a normal distribution curve.

This shape is shown as follows:

The shape of this graph is also typical of many other sets of data.

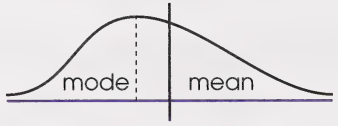
All sets of data having similar bell-shaped graphs are said to be **normally distributed**, or, they are called **normal distributions**.

Normal distribution curves have the appearance of a bell; hence the reason they are called **bell curves**. These curves are symmetrical about their means.

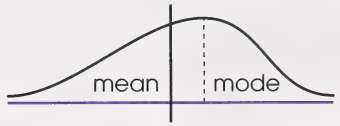
The following characteristics are present in all normal distributions:

- The mean, mode, and median are all at the middle of the graph. Under ideal conditions these measures of central tendency fall at the same point.

- A graph like the following is said to be skewed to the left. For this case, the mode is to the left of the mean.



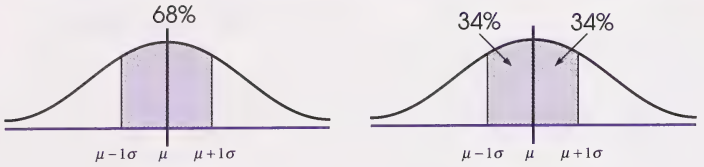
- The following graph is said to be skewed to the right. Here the mode is to the right of the mean.



In a normal distribution a certain percentage of the population always falls within 1, 2, or 3 standard deviation units from the mean.

- The percentage of the total area (the percentage of the total value of data) underneath the graph within one standard deviation of the mean is approximately 68%.

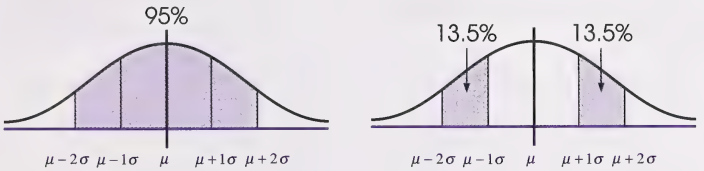
There is 34% on each side of the mean.



The set of data dealing with ski lengths was 71%, since it was a small set. As sets get larger, the data gets closer to 68%.

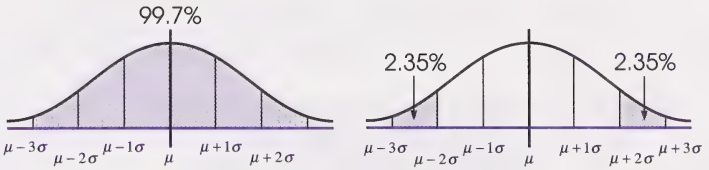
- The percentage of the total area (the percentage of the total value of data) within two standard deviations of the mean is 95%.

  There is approximately 13.5% outside one standard deviation and inside two standard deviations on each side of the mean.

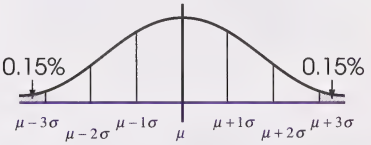


- The percentage of the total area (the percentage of the total value of data) within three standard deviations of the mean is about 99.7%.

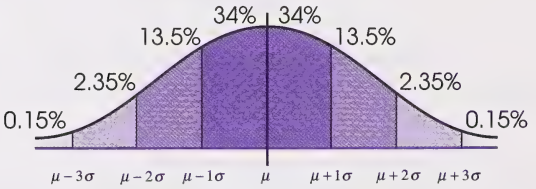
There is 2.35% outside two standard deviations and inside three standard deviations on each side of the mean.



- The percentage of the total area (the percentage of the total value of data) outside three standard deviations of the mean is 0.3%. There is 0.15% on each side of the mean.



The distributions are summarized on the following graph.



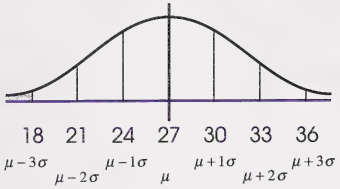
Note: When solving a problem that deals with normal distribution, you may find it helpful to make a drawing of this graph and replace the bottom labels with the information from the problem.

Example 1

A small electrical company is producing batteries that have an average life of 27 h and a standard deviation of 3 h. The company plans to replace any battery that does not last at least 18 h. If the company makes 200 000 batteries, how many batteries can it expect to replace? (Assume that this situation is normally distributed.)

Solution

Put this information on the normal distribution curve.



The area in which you are interested is to the left of the 18. This part of the graph is represented by 0.15% since it is more than 3 standard deviations from the mean. Therefore, the company will replace 0.15% of the batteries that it produces.

$$0.15\% \times 200\,000 = 0.0015 \times 200\,000$$
$$= 300$$

The company would expect to replace 300 batteries.

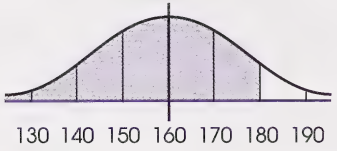
Now consider the following problem.

Example 2

A television station has decided to broadcast a local hockey game. From its statistics, the station realized that the average game is 160 minutes long with a standard deviation of ten minutes. How much time should it schedule for the game to be 97% sure that it will broadcast the entire game without disrupting normally scheduled programming?

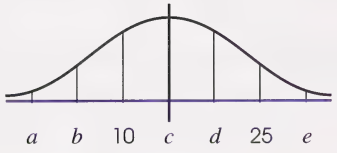
Solution

First, draw a normal distribution curve and enter the given information.



From the graph, you can see that approximately 97% of all games are completed in less than two standard deviations above the mean (50% below the mean and 4% above the mean). This means the station should allow 180 minutes ($160 \text{ min} + 2 \times 10 \text{ min}$) or three hours.

1. Complete the following graph.



2. A small company packages chocolates. The mean mass of the chocolates is 45 g with a standard deviation of 2 g.

   a. If the company produces 10 000 chocolates in a production run, how many of the chocolates will be within 2 g of the required mass?

   b. How many chocolates will be rejected in a production run of 100 000 if the mass must be between 43 g and 49 g?

3. A large appliance company has decided to boost its sales by offering a replacement guarantee. However, it does not want to replace more than 2.5% of the appliances that it manufactures. If the average life of the appliance is fifteen years with a standard deviation of two years, for what length of time should the company make the guarantee?

4. Determine the percentage of the normally distributed population that will lie between the following:

   a. μ and $\mu + \sigma$
   b. $\mu - 2\sigma$ and $\mu - 1\sigma$
   c. $\mu + 2\sigma$ and $\mu - 1\sigma$
   d. $\mu - 3\sigma$ and $\mu + 3\sigma$

Check your answers by turning to the Appendix.

Sets of data under ideal conditions having similar bell-shaped curves are normally distributed. Normally distributed data has particular interest to statisticians who wish to make predictions about the future based upon known data.

Activity 3: Applications of *z*-Scores

In the previous activity, you were able to compare data from different sets by finding the mean and the standard deviation, and then comparing how the data was distributed according to the number of standard deviations.

Using the number of standard deviations gave you a convenient measure that was used with all types of data that were normally distributed.

Up to this point you were able to work only with whole units of standard deviations. A system needs to be developed that will allow you to work with fractional units of standard deviation.

The units that you are going to use are called ***z*-scores**. Standard deviation units are used with z-scores.

One standard deviation to the right of the mean is a z-score of 1. Two standard deviations to the right of the mean is a z-score of 2. Three standard deviations to the right of the mean is a z-score of 3.

One standard deviation to the left of the mean is a z-score of -1.
Two standard deviations to the left of the mean is a z-score of -2.
Three standard deviations to the left of the mean is a z-score of -3.

A sign is assigned to the z-score to represent the direction from the mean. Positive is to the right of the mean and negative is to the left of the mean.

Use the following formula to find the z-score for any value in a set of data.

$$z\text{-score} = \frac{x - \mu}{\sigma}$$

In some situations, using the mean and the standard deviation with particular values of x will result in a fractional number for the z-score.

Start with two values, 11 and 17, from a set of data that has a mean of 12 and a standard deviation of 2. What are the z-scores for these two values?

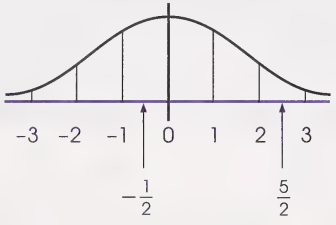
When $x = 11$,

$$z\text{-score} = \frac{x - \mu}{\sigma}$$
$$= \frac{11 - 12}{2}$$
$$= -\frac{1}{2}$$

When $x = 17$,

$$z\text{-score} = \frac{x - \mu}{\sigma}$$
$$= \frac{17 - 12}{2}$$
$$= \frac{5}{2}$$

Put these z-scores on the normal distribution curve.

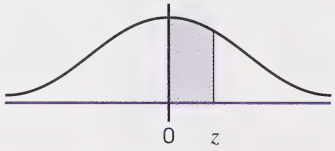


In Activity 2, the standard deviation was an intermediate tool that was used to find the percentage of the area under the graph. What are the areas for the z-scores?

Find the table (in the Appendix) labelled Standard Normal Distribution Table. This table will be used to find the areas.

How to Use the Standard Normal Distribution Table

At the top of the table is a normal distribution graph like the one shown here.



The numbers given in the table represent the area between the 0 (or mean) and the z-score. The numbers that represent the area are written as decimal numbers from 0 to 0.4990, where 0 represents no area under the graph and 0.4990 represents almost half of the area under the graph.

But how do you use the table to find the areas?

This table is designed to give you the areas for z-scores that are given to the second decimal place. Learn how to use this type of table by finding the area represented by the z-score 2.17. Follow these steps.

Step 1: Break the number into two addends.

$$2.17 = 2.1 + 0.07$$

The first addend is the ones and tenths of the z-score and the second addend is the hundredths of the z-score.

Step 2: Find the first addend, which is in the far left column of the table (z column). Move horizontally from this point.

Step 3: Find the second addend which is in the top row of the table. Move down from this point. Note that this addend is a single digit even though it represents hundredths. For example, 7 represents 0.07.

Step 4: Find the area where the two rectangles intersect. This value is the area for the z-score of 2.17, which is 0.4850.

z	0	1	2	3	4	5	6	7	8	9
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890

This number can easily be converted into percentage form.

$$0.4850 = 0.4850 \times 100\%$$
$$= 48.50\%$$

The same table can be used for finding the areas of negative z-scores. Since the normal distribution is symmetrical about the 0 or the mean, the area for a negative z-score is calculated in the same way.

For example, the area for a z-score of -1.17 is 0.3790, the same as for the z-score 1.17.

Look at the following examples.

Example 1

A group of 120 students took a physics test. How many students received a mark between 50% and 75% on the test if the mean was 50% and the standard deviation was 15%? Consider the distribution was normal.

Solution

First distinguish what z-scores you are looking for.

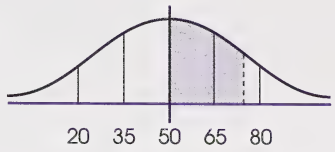
The mark of 50% is also the mean. This will be where you will start your measurement.

Note: The Standard Normal Distribution Table **always measures from the mean**.

You can ignore this mark (50%) for the time being.

The second mark, 75%, has the following z-score:

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{75 - 50}{15}$$
$$= \frac{25}{15}$$
$$\doteq 1.67$$



The z-score 1.67 is represented by the area 0.4525.

The product of this area and 120 (the total number of students) will give the number of students who scored between 50% and 75% on the test.

$$0.4525 \times 120 = 54.3$$
$$\doteq 54$$

Therefore, 54 students scored between 50% and 75% on the test.

Example 2

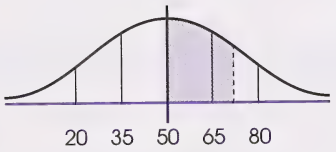
Using the information from Example 1, how many students scored between 40% and 72%?

Solution

This time you must find the area between 50% and 72%, and the area between 40% and 50%. The area that you are looking for will be the sum of these two areas.

Determine the area between 50% and 72%.

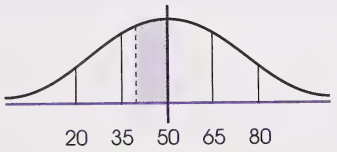
$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{72 - 50}{15}$$
$$= \frac{22}{15}$$
$$\doteq 1.47$$



The area for this z-score is 0.4292.

Now determine the area between 40% and 50%.

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{40 - 50}{15}$$
$$= \frac{-10}{15}$$
$$\doteq -0.67$$



The area for this z-score is 0.2486. (To look this z-score up in the table, ignore the negative sign.)

Now find the total area.

$$0.4292 + 0.2486 = 0.6778$$

Use the area to find the number of students.

$$0.6778 \times 120 = 81.336$$
$$\doteq 81$$

About 81 students received a mark between 40% and 72%.

Example 3

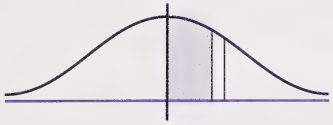
Using the information from Example 1 and Example 2, how many students received a mark between 72% and 75% on this physics test?

Solution

In this particular case, the area between 50% and 72% will not be part of the answer. This area must be removed from the area for 50% to 75%. This means that the area for 72% must be subtracted from the area for 75%.

The shaded area is the area you are looking for.

Locating the correct area under the normal distribution curve is crucial for solving the problem. If you want a more detailed explanation, turn to Section 3: Extra Help.



$$\text{Area of } 75\% - \text{Area of } 72\% = 0.4525 - 0.4292$$
$$= 0.0233$$

$$\therefore\ 0.0233 \times 120 = 2.796$$
$$\doteq 3$$

Therefore, 3 students received a mark between 72% and 75%.

1. What is the z-score for the distance of two standard deviations left of the mean?

2. The heights for ten basketball players are given. What are the z-scores for each height, assuming a normal distribution?

188, 190, 192, 196, 196, 198, 199, 202, 202, and 207

3. During a major-league career of 25 seasons, a baseball player averaged 150 hits a season with a standard deviation of 20 hits. In how many seasons did the player have at least 185 hits?

4. The expected working life of a refrigerator is normally distributed with a mean of seven years and a standard deviation of one year. In one year, the manufacturer sold 1200 refrigerators. Determine the number of refrigerators working longer than eight years. The manufacturer will replace any refrigerator that stops working when it is less than five years old. Determine the number of refrigerators that must be replaced.

5. An appliance distribution company knows that the average life of a certain toaster is 9.6 years and the standard deviation is 5.5 years. The company does not want to replace more than 9% of the toasters that are sold. What guarantee, to the nearest tenth of a year, should the company provide in order to satisfy its replacement policy?

6. From past records, a manufacturing company knows that its air compressors have an average working period of twelve years with a standard deviation of three years. What percentage of these compressors will have to be repaired under warranty if the manufacturer guarantees these compressors for eight years? Assume a normal distribution and give the percentage to the nearest whole number.

Check your answers by turning to the Appendix.

The z-scores determine the fractional units of standard deviation. These are important in answering questions based on real-life situations.

Activity 4: *z*-Scores and Probability

The meteorologist has forecasted a 20% probability of rain. Do you think it will rain?

What is probability?

Read the following different definitions:

- Probability is a branch of mathematics concerned with the study of the chance that a given event will occur.
- The probability of an event happening is the total number of favourable outcomes divided by the total number of outcomes where all of the outcomes have an equal chance of happening.
- The probability of an event is a real number between 0 and 1 inclusive that indicates how likely it is that an event will happen.

Take a look at a simple situation to help you understand this information.

A die has six faces with dots that represent the numbers 1 through 6. Therefore, all of the possible outcomes of rolling a die are 1, 2, 3, 4, 5, and 6. All of the possible outcomes of the experiment are referred to as the **sample space**.

The **outcomes** are all of the different results that can happen. In this case, there are six different outcomes. All of the outcomes are the sample space and they all must have the same chance to occur.

The **event** is made up of all of the favourable outcomes. In this case, you are looking at the event that a five will be rolled on the die.

The **favourable** or **desirable outcomes** are the results that you want to appear. In this case, the number five is the only favourable outcome.

According to the second definition,

$$\text{probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$P(\textit{roll a five}) = \frac{1}{6}$$

Now you can use this equation to make a statement.

The chance of rolling a five on a die is one out of six.

As the first definition says, a probability statement is making a prediction about the future. If you rolled a die, there is one chance out of six that a five will come up, or if the die is rolled six times, a five will come up once.

The third definition puts a limit on the type of numbers that can be used for probability. As you have already seen, probability gives you a rational or fractional number. The last definition puts a lower limit of 0 and an upper limit of 1 on the number used for probability.

Make a closer examination of these two cases.

What is the chance of getting a seven on a die?

$$\text{probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$P(\text{roll a seven}) = \frac{0}{6}$$
$$= 0$$

This case examines the situation where there is no chance of the event happening. This situation is referred to as an **impossible event**. There is no way you can get a smaller probability than 0 from this formula.

What is the chance of getting a natural number that is less than seven on the roll of a die?

$$\text{probability of an event} = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$P(< \text{seven}) = \frac{6}{6}$$
$$= 1$$

This case examines the situation where the event must happen. This situation is referred to as a **certain event**. Since all of the outcomes are also the favourable outcomes, it is not possible to get a probability higher than 1.

Another method for finding the probability of a situation is to use z-scores.

The areas in the Standard Normal Distribution Table are already set up to represent probabilities. The table considers the entire area under the graph to be 1. Each of the areas in the table will then represent the probability (that is, between the mean and the stated z-score) of an event happening.

See how this is done through an example.

Example

The mean duration of a common cold is ten days with a standard deviation of two days. What is the probability that a cold will last longer than thirteen days?

Solution

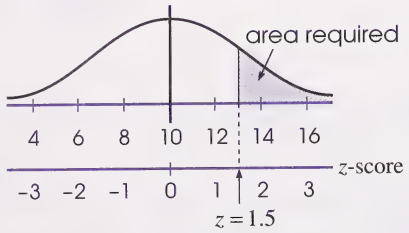
First use the z-score formula.

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{13 - 10}{2}$$
$$= 1.5$$

The area associated with a z-score of 1.50 is 0.4332.

This area is the amount between the mean and the z-score. According to the question, you are looking for the area associated with the z-score. Therefore, to find this required area, the area found in the table must be subtracted from 0.5 (the area for half of the graph).

$0.5 - 0.4332 = 0.0668$



The probability of having a common cold longer than thirteen days is 0.0668.

1. Find the probability of getting the following z-scores.

a. $P(z \leq 0)$
b. $P(0 \leq z \leq 2)$
c. $P(z \geq 1.80)$
d. $P(-0.89 \leq z \leq 1.24)$
e. $P(-2.37 \leq z \leq -0.92)$

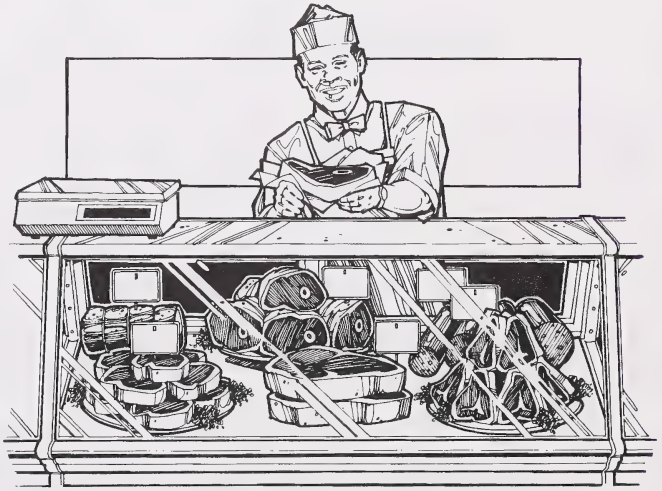
2. The average life expectancy of a cat is thirteen years with a standard deviation of 1.3 years. Determine the probability that a cat will live for the following periods of time.

a. at least fifteen years
b. at least twelve years
c. ten years or less
d. fourteen years or less

3. A firm manufactures batteries that have a mean life of 35 h and a standard deviation of 6.5 h. Determine the probability that a battery will last for the following periods of time.

a. 32 h or less
b. 43 h or less
c. at least 47 h
d. at least 33 h

4. In a supermarket, black forest ham is in the cooler three days on average before it is purchased. If there is a probability of 0.0179 of selecting a package which has been sitting on the shelf for 1.3 d or less, what is the standard deviation of the shelf period of the black forest ham?



5. Use the following table to determine the probability, to two decimal places, that a fluorescent light tube will last at least 820.5 h.

Lifetime of 200 Fluorescent Tubes	
Lifetime (h)	Frequency
600.5–650.5	12
650.5–700.5	18
700.5–750.5	20
750.5–800.5	39
800.5–850.5	68
850.5–900.5	27
900.5–950.5	16

6. A manufacturer of thumbtacks determines the mean number is 103 and the standard deviation is 2 for a box of thumbtacks advertised as containing 100 tacks. Determine the probability that a box will contain 100 or more tacks. Assume a normal distribution.

Check your answers by turning to the Appendix.

In this activity, you had an opportunity to apply z-scores to solve problems involving probability distributions.

Activity 5: Preparing for the Diploma Exam

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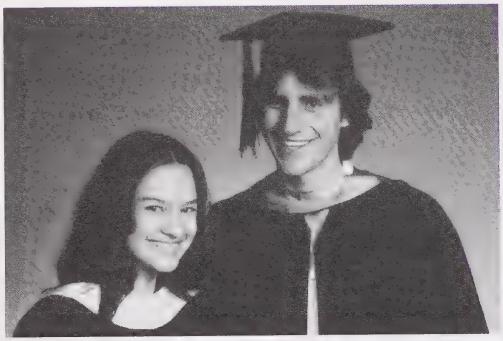
You have worked hard to get to this point in the course. Congratulations! You have covered a great deal of material and deserve the opportunity to demonstrate how much you have learned and the skills you have developed. Are you ready for the diploma exam?

This activity will help you to prepare to write the Mathematics 30 Diploma Exam. You also have a final test before the diploma exam; you may need some pointers to prepare for both exams.

It is important that you go into the diploma exam with a positive attitude. You have to be prepared to give your personal best. After all, your performance in this course and on this exam will most likely be important to your future aspirations. It is time for you to pull all your mathematics knowledge together in an organized manner so that you will be well prepared to meet the challenge that lies ahead.



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In the Module Overview and Appendix of Module 1 of this course, you were given several pages of important information that will help you prepare for the diploma exam. Review the sections Preparing to Write the Mathematics 30 Diploma Exam—What you should do before the exam! and Suggestions for Students Writing the Mathematics 30 Diploma Exam—What you should know when writing the exam! in the Appendix of Module 1; then answer the following questions.

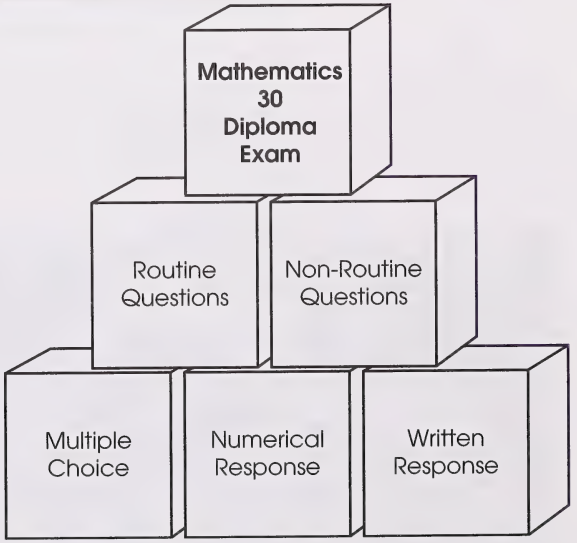
1. List five things you can do to prepare yourself for writing the diploma examination.

2. Identify five strategies for answering multiple-choice questions.

3. Identify three stategies for answering written-response questions.

Check your answers by turning to the Appendix.

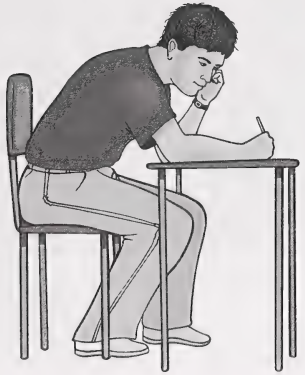
Diploma Examination Design



Constructing a good diploma exam is not an easy task! Several factors go into the design and format of the exam. Please note that the following descriptions are general in nature and may change somewhat from year to year.

You will be given 2.5 h to write the Mathematics 30 Diploma Exam. You should try to pace yourself to write the exam in about 2 h so that you have some extra time to review your answers.

As you discovered in the Module Overview in Module 1, there are three basic types of questions on the exam. (Refer to the chart.)



Diploma Examination Format			
Type of Question	**Description**	**Approximate Number of Questions**	**Approximate Percentage of Exam**
Multiple Choice	• all ranges of difficulty • some calculation questions • only one possible answer • machine-scored	40	57
Numerical Response	• may be more difficult • calculation or ordering questions • only the final answer written • machine-scored	9	13
Written Response	• complete answers written • concepts from more than one unit covered • communication skills marked • teacher-marked	4	30

In the future, the exam format may change. For example, for the written response component, there may be one essay-style question that covers all concepts.

Within the different types of questions already mentioned, you will be expected to use different levels of thinking, as shown in the following chart:

Diploma Examination Processes

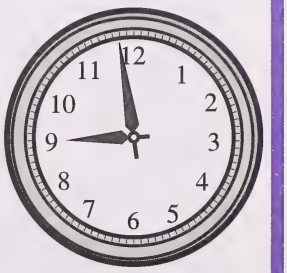
Type of Question	Description of Thinking Process	Approximate Percentage of Exam
Recall (Knowledge)	• remembering facts, descriptions, and explanations	25
Routine Problems and Questions (Application)	• applying concepts to problem solving • working in familiar areas	50
Non-Routine Problems (Higher Mental Activity)	• synthesizing concepts from more than one unit or module • applying your knowledge to new and novel situations	25

4. Considering the information in the Diploma Examination Processes chart, would it be worthwhile for you to spend most of your exam preparation time memorizing? Explain.

5. Review the Diploma Examination Format chart. Which format of question will likely require the most thoughtful and creative response?

Check your answers by turning to the Appendix.

Managing Time



With 2.5 hours in which to write the Mathematics 30 Diploma Exam, you usually don't have much time to sit and daydream. If you consider that there are 70 marks on the examination, the time available works out to a little over two minutes per mark. Keep in mind though, that you will probably need more time than this to be thoughtful and creative in the written-response section.

Considering the various restricting factors, the following chart is a suggested time budget. You know your strengths and weaknesses best, so you may adjust the time allotments accordingly.

	Total Number of Marks	Time per Mark (min)	Total Time (min)
Multiple Choice	42	about 2	84
Numerical Response	7	about 2	14
Written Response	21	about 2.5	52.5
Total	70	—	150.5

Note: The times given in the preceding chart include review time near the end of the exam.

6. What do you think is the best strategy to follow if you get stuck on a multiple-choice or a numerical-response question and you have already spent five minutes on it?

7. Do you think you have to answer the multiple-choice questions first, then the numerical response, then the written response, in the order that the exam is constructed? Explain.

Check your answers by turning to the Appendix.

Studying Techniques

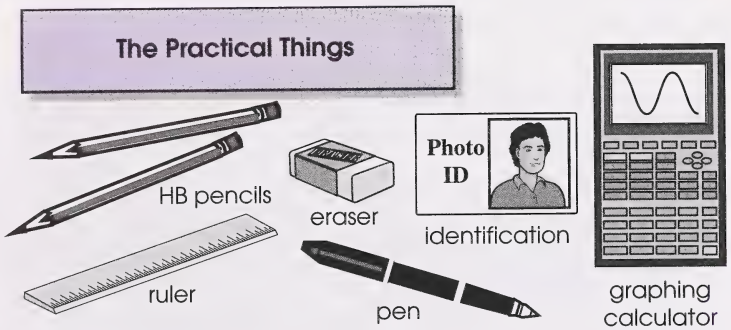
It is worth making a few points about study habits in your preparation for the diploma exam:

- **Be active when you study.** Don't spend long hours just sitting and reading your textbook or notebook. This is not an effective preparation method. Study actively; make point-form summaries or reattempt problems from your module booklets.

- **Anticipate typical questions and try to answer them.** Using the design of the exam given earlier in this activity, you know that the focus of the exam will be on things such as the big, interconnecting ideas, math skills, and applications to technology and society. Identify these kinds of questions from each module. Your assignments from each section are an excellent source of exam-type questions.

- **Work within your attention span.** You should know by now how long you can sit and study effectively. Break up your time between study, household chores, and recreation. Get into the habit of setting aside a certain time each day for study.

- **Stay healthy.** Studying the whole night before an exam, skipping meals, and eating junk food are definitely not good strategies! Your mind needs your body to be ready for the exam, too.



Here are a few final points about the material you take into the examination:

- Make sure that you have identification that has your photograph on it, such as a school card or driver's licence.
- Check your calculator batteries to make sure that your calculator will last through the entire exam. It is a good idea to have a spare set of batteries.
- Make sure that your calculator is "legal" to take into the exam. Contact your local high school or talk with your teacher or learning facilitator to find out the latest requirements for calculators in exams.
- HB pencils are a must for the multiple-choice and numerical-response answer sheets. Having more than one would be a good strategy.

Go into the exam with a positive attitude. You have come too far not to succeed now. Be prepared!

You Can Do This! Good Luck!

Follow-up Activities

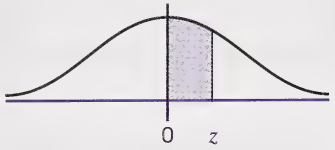
If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You may find it helpful to view the video titled *Graphing Statistics*, from the *Graphing Mathematical Concepts* series, ACCESS Network. You may also view the video titled *Statistics*, from the *Catch 30* series, ACCESS Network. Pay attention only to the Normal Distribution Curve. Both programs will help to reinforce the concepts of standard deviation, normal distribution, z-scores, and probability. These videos are available from the Learning Resources Distributing Centre.

Selecting the correct area under the normal distribution graph is crucial to solving problems. You will take a closer look at how to select the appropriate area(s) under the graph and the procedure that must be followed to solve for that area.

A Standard Normal Distribution Table, as given in the Appendix, always measures the area to the right of the mean. On the normal distribution curve, this is always to the right of the zero z-score, as shown in the following graph.



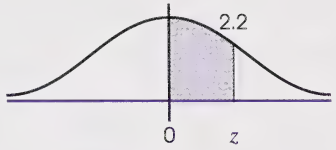
Since the table will only give areas to the right of the z-score, the largest possible area that you can have is 0.5 or half of the area under the graph.

Take a look at how to find each of the types of areas using the graph and the table.

Type 1: Zero z-Score to Positive z-Score

The area to be found is the area from the mean (or the 0 z-score) to a positive z-score. Since the table is already set up for finding this type of area, you can just read the area from the table.

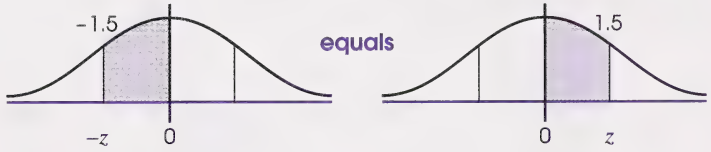
For example, from the table, $z = 2.2$ gives an area of 0.4861.



Type 2: Zero z-Score to Negative z-Score

This area is from the mean to a negative z-score. The table only gives areas to the right of the mean. However, since the normal distribution curve is symmetrical, a specified distance left of the mean will yield the same area as that distance right of the mean.

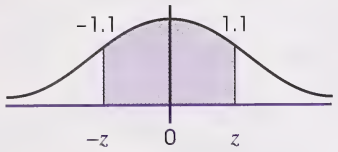
To find this area, ignore the negative sign on the z-score and look up the area in the chart as if you were looking for the area for a positive z-score.



If $z = -1.5$, read $z = 1.5$ in the table. This gives an area of 0.4332.

Type 3: Negative to Positive z-Score

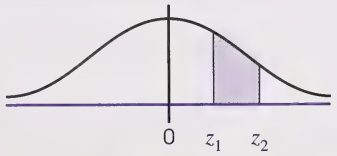
This type of area is broken up into two different areas: an area from the zero z-score to the positive z-score and an area from the zero z-score to the negative z-score. For $z = 1.1$, the area is 0.3643; and for $z = -1.1$, the area is also 0.3643. The total area is $2(0.3643)$ or $(0.3643 + 0.3643) = 0.7286$.



The following diagrams also illustrate negative to positive z-score areas.

The total area is found by adding the two individual areas for each distributive curve.

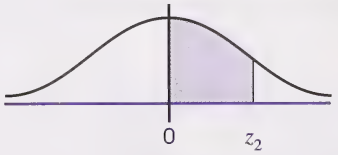
Type 4: Positive to Positive z-Score



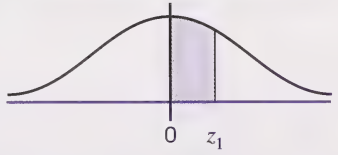
This type of area is also broken up into two different areas. One area will be from the zero z-score to the higher positive z-score (z_2-score) and the other area will be from the zero z-score to the lower positive z-score (z_1-score).

The two areas that you will be dealing with are shown here. The area is found by taking the difference of the two areas.

Step 1: Take this area.



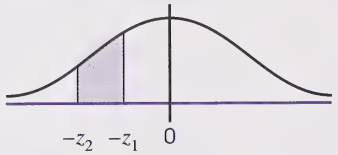
Step 2: Subtract this area.



If $z_2 = 2.8$, the area is 0.4974. If $z_1 = 0.9$, the area is 0.3159.

$$\begin{aligned} \text{area required} &= z_2 - z_1 \\ &= 0.4974 - 0.3159 \\ &= 0.1815 \end{aligned}$$

Type 5: Negative to Negative z-Score

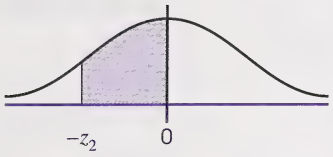


This type of area is found using the same procedure that was used for the positive to positive z-score area.

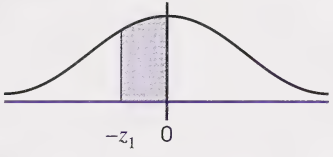
The area is broken up into two areas. One area is from the zero z-score to the smaller negative z-score (z_2-score) and the other area is from the zero z-score to the larger negative z-score (z_1-score).

Ignore the negative signs on the z-scores. Look up the areas in the table as if you were looking for the positive z-scores.

Step 1: Take this area.



Step 2: Subtract this area.

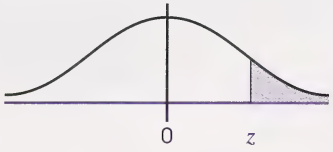


If $z_2 = -3.1$, the area is 0.4990. If $z_1 = -1.4$, the area is 0.4192.

$$\text{required area} = 0.4990 - 0.4192$$
$$= 0.0798$$

Type 6: Positive Extreme Area

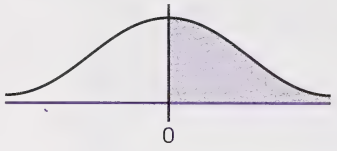
This area is on the extreme right side of a positive z-score.



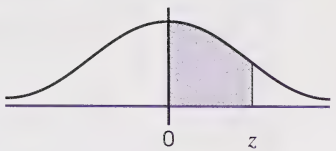
You know that half of the graph has an area of 0.5. Therefore, to find this extreme area, find the area from the zero z-score to the positive z-score.

You then subtract this area from 0.5.

Step 1: Take this area (0.5).

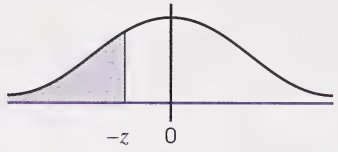


Step 2: Subtract this area.



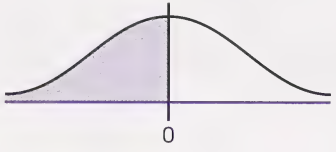
Type 7: Negative Extreme Area

This area is on the extreme left side of a negative z-score. Ignore the negative sign on the z-score. Use the method for finding a positive extreme area.

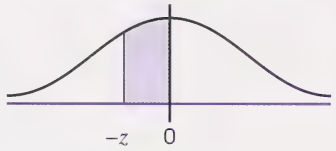


Half of the graph has an area of 0.5. Find the area from the zero z-score to the negative z-score. Subtract this area from 0.5.

Step 1: Take this area (0.5).



Step 2: Subtract this area.



All other types of graphs are made up of combinations of these seven types. In those cases, break the graphs down into the individual types and then find the areas.

1. Identify the type or types of area for each of the following curves.

a.

b.

c.

d.

e.

f.

2. Identify the type or types of area for each of the following. It may help to first sketch the graph.

a. $0 \le z \le 2.76$

b. $-2.82 \le z \le -1.09$

c. $z \le -2.3$

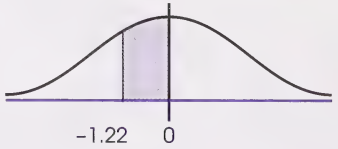
d. $z \le -2$ and $z \ge 2$

e. $2.3 \le z \le 2.32$

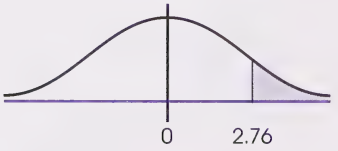
f. $-0.04 \le z \le 1.03$

3. Find the areas of the following.

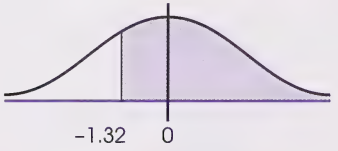
a.



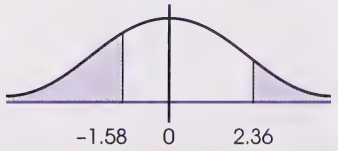
b.



c.



d.



4. Find the following areas.

a. $z \leq -1.02$ **b.** $z \geq -2.04$

c. $0 \leq z \leq 2.45$ **d.** $-2.38 \leq z \leq -1.89$

Check your answers by turning to the Appendix.

Enrichment

You can use a calculator with statistical functions in order to find the standard deviation of a set of **grouped data**. (You should refer to the operator's manual to learn the proper way to use your calculator.)

In the following table, the frequency represents the number of days that a certain range of requests for steel pipes, 180 cm long, were made. The class limits state the range of the number of requests on any particular day. The class midpoint is the average of the class limits $\left(\text{e.g., } \frac{50+59}{2} = 54.5\right)$.

Class	Class Limits	Frequency (days)	Class Midpoint
1	50–59	22	54.5
2	60–69	34	64.5
3	70–79	41	74.5
4	80–89	54	84.5
5	90–99	43	94.5
6	100–109	38	104.5
7	110–119	18	114.5

The procedure to find the standard deviation σ_n and mean $\bar{x}$ for a grouped data using the CASIO *fx-7700G* calculator is as follows:

Step 1: Turn on the calculator. Press Mode ×.

Run/SD will appear in the first row of the display to inform you that the calculator is in its statistical mode.

Step 2: Press Mode Shift 2.

Step 3: Press Shift 3 F2.

Scl will appear on the upper left corner of the display.

Step 4: Press the EXE PRE PRE.

The number **0.** will be displayed.

Step 5: Enter the data. For example, to enter the first two entries (54.5 and 22, and 64.5 and 34) do the following:

5 4 • 5 F3 2 2 F1

6 4 • 5 F3 3 4 F1

Scl	0.
54.5; 22	54.5
64.5; 34	64.5

Now enter the remaining data.

Step 6: Press F5 F3 EXE.

The number **250.** is displayed for n.

Step 7: Press PRE F4 F1 EXE.

The mean $\bar{x} =$ **84.42** is displayed.

Step 8: Press F3 EXE.

The standard deviation $x\sigma_{n-1} =$ **17.145 538 13** appears on the screen.

Use a graphing calculator for questions 1 and 2.

1. Find the mean and the standard deviation for the following grouped data.

Class	Class Limits	Frequency	Class Midpoint
1	12–20	4	16
2	21–29	12	25
3	30–38	32	34
4	39–47	76	43
5	48–56	59	52
6	57–65	18	61
7	66–74	11	70
8	75–83	3	79

2. Find the mean and the standard deviation for the following grouped data. (**Hint:** Find the class midpoint.)

Class	Class Limits	Frequency
1	22–26	8
2	27–31	10
3	32–36	29
4	37–41	57
5	42–46	73
6	47–51	43
7	52–56	22
8	57–61	9

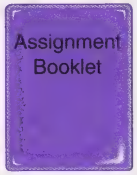
Check your answers by turning to the Appendix.

Conclusion

In this section you studied standard normal distributions. You analysed these distributions in terms of their means (or averages) and their standard deviations. You also determined percentages or proportions of the population, described by a given distribution, that fell within well-defined categories. To determine these percentages, you calculated measures called z-scores by using the mean and standard deviation, by referring to z-score tables, or by using a scientific calculator. The analysis in this section is an important aspect of the interpretation of statistical data.

Statistical information is a critical commodity in society. Manufacturers, for instance, as part of good business practice, collect data on the goods they produce. Decisions on warrantee periods and guarantees directly affect sales and profit margin. These decisions cannot be made arbitrarily or the manufacturer would not remain in business for very long.

Assignment



You are now ready to complete the section assignment.

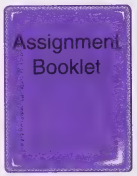
Module Summary

This module dealt with two branches of mathematics: combinatorics—the study of permutations and combinations—and the part of statistics relating to the standard normal distribution or bell curve.

Specifically, in Section 1 you dealt with analysing the number of ways objects can be arranged in a row, or in a circle, or on a ring. In Section 2 you explored the number of ways objects can be selected. As well, you used the concept of selections to develop the Binomial Theorem—a process of expanding powers of binomials. In Section 3 you dealt with the branch of statistics that looks at how data is distributed. In this section, you used the concepts of mean, standard deviation, and z-score to analyse those distributions.

The people in the photograph are seated in just one of 5! or 120 possible arrangements or permutations. If the man standing in the picture were to sit on the bench with the other five people, he would select one of six possible positions. He could choose to sit at either end or in one of four possible spots on the bench between two other people. The decision where to sit is a combination. When seated, all six people could be rearranged in 720 different ways! Did you think that resting on a bench would be that complicated?

Final Module Assignment



You are now ready to complete the final module assignment.

COURSE SURVEY FOR MATHEMATICS 30

Please evaluate this course and return this survey with your final module assignment. This course is designed in a new distance learning format, so we are interested in your responses. Your constructive comments will be greatly appreciated, as future course revisions can then incorporate any necessary improvements.

Name _____ Age ❑ under 19 ❑ 19 to 40 ❑ over 40

Address _____ File No. _____

_____ Date _____

_____

Design

1. This course contains a series of student module booklets. Do you like the idea of separate booklets?

_____

_____

2. Have you ever enrolled in a correspondence or distance learning course that arrived as one large volume?

❑ Yes ❑ No If yes, which style do you prefer?

_____

_____

3. The student module booklets contain a variety of self-assessed activities. Did you find it helpful to be able to check your work and have immediate feedback?

❑ Yes ❑ No If yes, explain.

_____

_____

4. Were the questions and directions easy to understand?

❑ Yes ❑ No If no, explain.

_____

_____

5. Each section contains follow-up activities. Which type of follow-up activity did you choose?

   ❑ mainly extra help
   ❑ mainly enrichment
   ❑ a variety
   ❑ none

   Did you find these activities beneficial?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

6. Did you understand what was expected in the assignment booklets?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

7. The course materials were designed to be completed by students working independently at a distance. Were you always aware of what you had to do?

   ❑ Yes ❑ No If no, provide details.

   _____

   _____

8. This course may include drawings, photographs, and charts.

   a. Did you find it helpful to have these visuals?

      ❑ Yes ❑ No Comment on the lines below.

      _____

      _____

   b. Did you find the variety of visuals motivating?

      ❑ Yes ❑ No Comment on the lines below.

      _____

      _____

9. Suggestions for audiocassette and videocassette activities may have been included in the course. Did you make use of these media options?

   ❑ Yes ❑ No Comment on the lines below.

   _____

   _____

Only students enrolled in a Junior High course need to complete the following question.

10. Students are often directed to work with their learning facilitator. How well did you work as a team?

    Student's comments: _____

    _____

    _____

    _____

    Learning Facilitator's comments: _____

    _____

    _____

    _____

Course Content

1. Was enough detailed information provided to help you learn the expected skills and ojectives?

   ❑ Yes ❑ No Comment on the lines below.

   _____

   _____

2. Did you find the work load reasonable?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

3. Did you have any difficulty with the reading level?

❑ Yes ❑ No Please comment.

4. How would you assess your general reading level?

❑ poor reader ❑ average reader ❑ good reader

5. Was the material presented clearly and with sufficient depth?

❑ Yes ❑ No If no, explain.

General

1. What did you like least about the course?

2. What did you like most about the course?

Additional Comments

Only students enrolled with the Alberta Distance Learning Centre need to complete the remaining questions.

1. Did you contact the Alberta Distance Learning Centre for help or information while doing your course?

   ❑ Yes ❑ No If yes, approximately how many times? _____

   Did you find the staff helpful?

   ❑ Yes ❑ No If no, explain.

   _____

   _____

2. Were you able to fax any of your assignment response pages?

   ❑ Yes ❑ No If yes, comment on the value of being able to do this.

   _____

   _____

3. If you mailed your assignment response pages, how long did it take for their return?

   _____

   _____

4. Was the feedback you received from your correspondence or distance learning teacher helpful?

   ❑ Yes ❑ No Please comment.

   _____

   _____

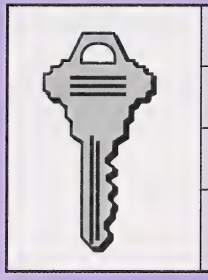
   _____

   _____

Thanks for taking the time to complete this survey. Your feedback is important to us. Please return this survey with your final module assignment.

Instructional Design and Development
Alberta Distance Learning Centre
Box 4000
Barrhead, Alberta
T7N 1P4

APPENDIX



Glossary

Diploma Exam Information

Suggested Answers

Standard Normal Distribution Table

Glossary

Binomial Theorem: a rule for the expansion of a power of a binomial

Circular permutation: arrangement of objects around a circle

Combination: selection of different objects where order is not important

Combinatorics: the study of permutations and combinations

Factorial: for a positive integer n, the product of all the positive integers less than or equal to n

Linear permutation: arrangement of things along a straight line in a definite order

Mean: a single number representing a set of numbers

Normal distributions: sets of data with a bell-shaped curve

Pascal's Triangle: a triangular array of numbers composed of the coefficients in the expansion of $(x+y)^n$

Permutation: an ordered arrangement or sequence of all or part of a set of things

Ring permutation: arrangement of objects in circular permutation with reflections

Standard deviation: a measure that describes the distribution of data about the mean

Statistics: collection of data or methods of planning experiments for obtaining data and drawing conclusions or making decisions on the basis of available data

Diploma Exam Information

Preparing to Write the Mathematics 30 Diploma Exam—What you should do before the examination!

1. Prepare a course schedule.

   - Design your schedule for the two-week period (minimum) before the exam.
   - Divide the course material into sections and indicate on the schedule the time blocks to be devoted to each section.
     - Take into account the exam blueprint available from your teacher (*Diploma Examinations Program Information Bulletin* for the course). This blueprint indicates the relative importance (weighting) of each unit in the course. Note that course units are not equally weighted on the diploma exam.
     - Take into account modules/concepts that you find most difficult, i.e., allocate more time for the review of these.

2. Obtain and review examination schedules, rules, and policies.

    - Record the time and place of writing.
    - Note minimum and maximum writing times permitted.
    - Prepare to remain in the examination room for 2.5 h (tissues, cough drops, and so on).
    - Identify materials allowed for writing each exam, such as pencils, pens, calculators, mathematical instruments, and a clear plastic ruler.

3. Identify, collect, and practise examples of each type of question that will be asked.

    - Review the format of previous diploma exams.
    - Learn the meanings of the key "directing" words.

4. Use memory aids such as the following:

    - Colour code, underline, highlight, or jot key words in margins.
    - Number points to be memorized.
    - Group word and idea associations.
    - Read aloud key words, expressing key words in your own words.

Suggestions for Students Writing the Mathematics 30 Diploma Exam—What you should know when writing the examination!

1. Make use of the formula sheet included with the diploma exam.

2. Do not be afraid to answer each question even if you are not sure of the correct solution to the problem. A penalty is NOT given for guessing on the machine-scored section of the exam. Partial marks are often awarded for incomplete answers in the written-response section of the exam.

3. If you are stuck on a question, mark the alternatives that you know are incorrect and choose from the ones that are left using logical guessing strategy. Think of the questions as challenges, and cultivate a positive attitude about your ability to answer them.

4. Scan the written-response, multiple-choice, and numerical-response sections of the examination before answering the questions. A question in one section of the examination may jog your memory about a question in another section.

5. When first reading a multiple-choice question, locate and circle key words to help clarify the meaning of the question. Then hide the alternatives and try to formulate an answer of your own. This way you will not be distracted by the alternatives.

6. Have a good reason for changing an answer. Do not change an answer on a hunch. Do not waste your time looking for patterns of As, Bs, Cs, or Ds in multiple-choice answers. There are none.

7. You may not have time to write and edit a complete rough copy for each written-response question, but you should prepare an outline of your answer and use it as a guide when writing your good copy.

8. When completing a written-response question, keep in mind the reader of your response. The reader will want to know how well you

   - understand the problem or the mathematical concept
   - can correctly use the mathematics involved
   - can use problem-solving strategies and explain your answer and procedures
   - can communicate your solutions and mathematical ideas

9. Rewriting a statement of the question is often a good way to begin a written response. Conclude with a summary statement. Be sure you have clearly explained all assumptions and have verified your conclusions.

10. Keep track of the time and pace yourself. Put a check mark by items that you are uncertain about and return to them if there is time at the end of the exam.

Suggested Answers

Section 1: Activity 1

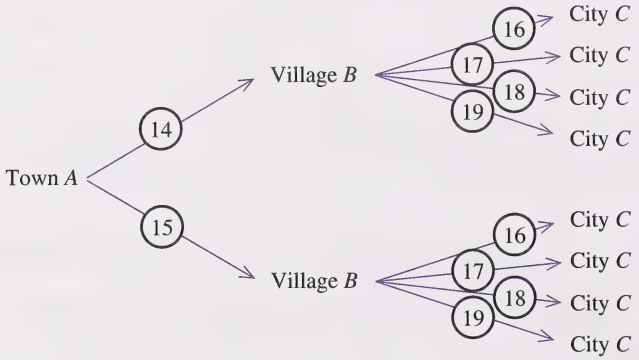
1. $4 \times 3 = 12$

The appliance manufacturer can make 12 different ranges.

2. $6 \times 5 \times 2 = 60$

The dealer can order 60 cars.

3.



4. a. $8 \times 7 = 56$

Peter can enter and leave in 56 ways.

b. $15 \times 15 = 225$

Peter can enter and leave the stadium in 225 ways.

5. Letter 1 can go into any one of the two mailboxes. Letter 2 can go into any one of the two mailboxes, and so on.

$$\therefore \text{ number of ways} = 2 \times 2 \times 2 \times 2 \times 2$$
$$= 2^5$$
$$= 32$$

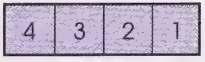
The letters can be mailed in 32 ways.

6. There are three mailboxes and n letters.

$$\therefore \text{ number of choices} = 3^n$$
$$3^n = 243$$
$$3^n = 3^5$$
$$n = 5$$

Therefore, Wendy has five letters.

7. Four boxes are used to represent the number of choices for each letter.



There are four choices for the first box.

There are three choices for the second box since one letter is used in box one.

There are two choices for the third box since boxes one and two have used two letters.

Finally, the remaining letter goes into box four.

The total number of four-letter words is $4 \times 3 \times 2 \times 1 = 24$.

8.

Box one has one choice, the letter C.

Box two has four choices since there are four remaining letters.

Box three has three choices since boxes one and two have used two letters.

Box four has two choices from the remaining two letters.

Finally, box five has the only remaining choice.

The total number of five-letter words is $1 \times 4 \times 3 \times 2 \times 1 = 24$.

9.

Boxes one and four have 1 choice each.

Box two has four choices since one L and one E are filled and M is restricted. (The remaining four letters are S-U-B-R.)

Box three has four choices also since an L and E and another letter are filled.

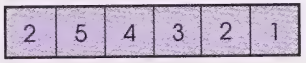
Box five has three choices from the remainig three letters.

Box six has two choices from the remaining two letters.

Box seven has one choice since one letter remains.

The total number of arrangements is $1 \times 4 \times 4 \times 1 \times 3 \times 2 \times 1 = 96$.

10. The two vowels are U and E. Thus, there are two choices for the first letter.



$2 \times 5 \times 4 \times 3 \times 2 \times 1 = 240$ words

11. **a.** No, it cannot since using the digits 5, 7, and 8 will produce numbers greater than 400.

b. The first digit can be chosen in **2** ways.

c. The second digit can be chosen in **5** ways.

d. The third digit can be chosen in **5** ways.

e. Apply the fundamental counting principle. The number of three-digit numbers is **50**.

12. The following three boxes are used to represent the number of choices for each digit in a three-digit odd number.

Deal with restrictions first when doing these problems.

First Digit	Second Digit	Third Digit
2	3	3
One of 2 or 4 can go here; thus, there are two choices.	There are three choices for this digit after the choices are made for the first and third digits.	One of 1, 7, or 9 can go here; thus, there are three choices.

Since the three-digit odd numbers are between 200 and 600, this means that a 2 or a 4 can be used for the first digit.

Three-digit odd numbers are required; thus, the third digit must end in a 1, 7, or 9. This means that there are three choices for the third digit.

When the choice of numbers has been determined for the first and third digits, there are three choices for the second digit since digits cannot be repeated. For example, if a 4 is used for the first digit and a 7 is used for the third digit, three choices (1, 2, or 9) can be used for the second digit. The total number of three-digit odd numbers is $2 \times 3 \times 3 = 18$.

13. The first three boxes are used to represent the number of choices for the three letters. The next three boxes represent the number of choices for the three digits.

First Letter	Second Letter	Third Letter
26	25	24

First Digit	Second Digit	Third Digit
9	8	7

There are twenty-six choices for the first letter since there are twenty-six letters in the alphabet. Since letters cannot be repeated, like AAB, there are twenty-five choices for the second letter. The third letter has twenty-four choices since repetitions are not allowed.

There are nine choices for the first digit. Digits cannot be repeated; thus, there are eight choices for the second digit and seven choices for the third digit.

Each of the letter choices can be matched with each of the digit choices.

$$\therefore \text{ number of licence plates} = (26\times25\times24)\times(9\times8\times7)$$
$$= 15\,600\times504$$
$$= 7\,862\,400$$

Therefore, the number of licence plates that can be made is 7 862 400.

Note: If both letters and digits could be repeated, the number of licence plates would be $26\times26\times26\times9\times9\times9 = 12\,812\,904$.

Section 1: Activity 2

1. $6! = 720$

The letters can be arranged in 720 ways.

2. $5! = 120$

Jerritt can read the magazines in 120 different orders.

3. $\dfrac{10!}{7!2!} = \dfrac{10\times9\times8\times\cancel{7!}}{2\times1\times\cancel{7!}}$

$= 360$

4. $\dfrac{4!}{5!2!} + \dfrac{6!}{3!2!} = \dfrac{\cancel{4!}}{5\times\cancel{4!}\times2!} + \dfrac{6\times5\times4\times\cancel{3!}}{\cancel{3!}\times2!}$

$= \dfrac{1}{5\times2\times1} + \dfrac{6\times5\times4}{2\times1}$

$= \dfrac{1}{10} + 60$

$= 60\dfrac{1}{10}$ or 60.1

5. $(k-2)(k-3)! = (k-2)!$

6. $\dfrac{(n+r-2)!}{(n+r-5)!} = \dfrac{(n+r-2)(n+r-3)(n+r-4)\cancel{(n+r-5)!}}{\cancel{(n+r-5)!}}$

$= (n+r-2)(n+r-3)(n+r-4)$

7. $(m+3)(m+2)!=(m+3)!$

8. a.
$$\frac{(n-3)!}{(n-4)!}=7$$
$$\frac{(n-3)(n-4)!}{(n-4)!}=7$$
$$(n-3)=7$$
$$n=10$$

b.
$$\frac{(n-1)!}{(n+1)!}=\frac{1}{6}$$
$$\frac{(n-1)!}{(n+1)(n)(n-1)!}=\frac{1}{6}$$
$$\frac{1}{(n+1)(n)}=\frac{1}{6}$$
$$n^2+n=6$$
$$n^2+n-6=0$$
$$(n+3)(n-2)=0$$
$$n+3=0 \quad \text{or} \quad n-2=0$$
$$n=-3 \qquad n=2$$

$n=-3$ is inadmissible since factorials of negative numbers are not allowed.

9. a.
$$\frac{(a+2)!}{a(a+2)}=\frac{(a+2)(a+1)(a)(a-1)!}{a(a+2)}$$
$$=(a+1)(a-1)!$$

b.
$$\frac{(n-r+2)!}{(n-r-2)!}$$
$$=\frac{(n-r+2)(n-r+1)(n-r)(n-r-1)(n-r-2)!}{(n-r-2)!}$$
$$=(n-r+2)(n-r+1)(n-r)(n-r-1)$$

c.
$$\frac{(a-1)!}{(a-3)!2!}=\frac{(a-1)(a-2)(a-3)!}{(a-3)!2!}$$
$$=\frac{(a-1)(a-2)}{2}$$

10. $\therefore\ n=2$

The four flags can be arranged in $4!$ ways.

$\therefore$ number of signals $= 4!$
$= 24$

A total of 24 different signals can be sent.

11. There are seven books on the shelf. The three biology books can be arranged in $3!$ ways.

The biology books are considered as one unit, so there are five items to be arranged. The number of arrangements is ${}_5P_5$.

$$\therefore \text{ number of arrangements} = 3! \times {}_5P_5$$
$$= 3! \times 5!$$
$$= 6 \times 120$$
$$= 720$$

The books can be arranged in 720 ways.

12. The six algebra books can be arranged in $6!$ ways.

The four geometry books can be arranged in $4!$ ways.

The geometry books are kept together in 2 positions (in front or back of the algebra books).

$$\therefore \text{ number of arrangements} = 6! \times 4! \times 2$$
$$= 720 \times 24 \times 2$$
$$= 34\,560$$

The books can be arranged in 34 560 ways.

Section 1: Activity 3

1. a. $${}_4P_3 + {}_5P_2 = \frac{4!}{(4-3)!} + \frac{5!}{(5-2)!}$$
$$= 4! + 5 \times 4$$
$$= 44$$

b. $$\frac{7!}{5!0!} = \frac{7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 1}$$
$$= 42$$

c. $${}_{10}P_3 \times {}_5P_5 = \frac{10!}{(10-3)!} \times \frac{5!}{(5-5)!}$$
$$= \frac{10!}{7!} \times \frac{5!}{0!}$$
$$= \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}} \times \frac{5!}{1}$$
$$= 720 \times 120$$
$$= 86\,400$$

2. The numbers can contain four digits, three digits, two digits, or one digit.

$$\therefore {}_5P_4 + {}_5P_3 + {}_5P_2 + {}_5P_1$$
$$= \frac{5!}{(5-4)!} + \frac{5!}{(5-3)!} + \frac{5!}{(5-2)!} + \frac{5!}{(5-1)!}$$
$$= 120 + 60 + 20 + 5$$
$$= 205$$

A total of 205 numbers can be formed.

3. You can hang five pictures, four pictures, three pictures, two pictures, or one picture.

$$\therefore {}_{10}P_5 + {}_{10}P_4 + {}_{10}P_3 + {}_{10}P_2 + {}_{10}P_1$$
$$= 30\,240 + 5040 + 720 + 90 + 10$$
$$= 36\,100$$

The pictures can be hung in 36 100 ways.

4. You can form words with 6 letters, 7 letters, or 8 letters.

$$\therefore\ {}_8P_6 + {}_8P_7 + {}_8P_8 = 20\,160 + 40\,320 + 40\,320$$
$$= 100\,800$$

A total of 100 800 words can be formed.

5. a. $\left(\frac{1}{2}\right){}_nP_7 = {}_{n-2}P_5$, where $n \in N$

$$\left(\frac{1}{2}\right)\frac{n!}{(n-7)!} = \frac{(n-2)!}{(n-2-5)!}$$
$$\frac{n!}{2(n-7)!} = \frac{(n-2)!}{(n-7)!}$$
$$\frac{n!}{2} = (n-2)!$$
$$\frac{n!}{(n-2)!} = 2$$
$$n(n-1) = 2$$
$$n^2 - n - 2 = 0$$
$$(n-2)(n+1) = 0$$

$n-2=0$ or $n+1=0$

$n=2$ $\quad$ $n=-1$

-1 is not a natural number.

Therefore, $n=2$.

b.
$$4 \times {}_nP_3 = 6 \times {}_{n-1}P_3$$
$$4 \times \frac{n!}{(n-3)!} = 6 \times \frac{(n-1)!}{(n-1-3)!}$$
$$\frac{4 \times n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = \frac{6 \times (n-1)(n-2)(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}}$$
$$4 \times n\cancel{(n-1)}\cancel{(n-2)} = 6 \times \cancel{(n-1)}\cancel{(n-2)}(n-3)$$
$$4n = 6(n-3)$$
$$4n = 6n - 18$$
$$-2n = -18$$
$$n = 9$$

6.

LS	RS
${}_nP_{r+2}$	$(n-r)(n-r-1) \times {}_nP_r$
$= \frac{n!}{(n-r-2)!}$	$= (n-r)(n-r-1) \times \frac{n!}{(n-r)!}$
	$= \cancel{(n-r)}\cancel{(n-r-1)}\frac{n!}{\cancel{(n-r)}\cancel{(n-r-1)}(n-r-2)!}$
	$= \frac{n!}{(n-r-2)!}$

LS = RS

7. The men may be seated in ${}_5P_5$ ways and the women may be seated in ${}_4P_4$ ways. Each arrangement of the men may be associated with each arrangement of the women.

$$\begin{aligned}\therefore \text{ number of arrangements} &= {}_5P_5 \times {}_4P_4 \\ &= 5! \times 4! \\ &= 2880\end{aligned}$$

The people can be arranged in 2880 ways.

8. The adults may stand in ${}_3P_3$ ways and the children may stand in ${}_4P_4$ ways. The children can occupy four different positions (child, adult, child, adult, child, adult, child)

$$\begin{aligned}\therefore \text{ number of arrangements} &= 4 \times {}_3P_3 \times {}_4P_4 \\ &= 4 \times 3! \times 4! \\ &= 576\end{aligned}$$

The adults and children can be arranged in 576 ways.

9. $$\begin{aligned}{}_{26}P_2 \times {}_{10}P_4 &= \frac{26!}{(26-2)!} \times \frac{10!}{(10-4)!} \\ &= 650 \times 5040 \\ &= 3\,276\,000\end{aligned}$$

Therefore, there are 3 276 000 possible serial numbers.

10. The first digit can be 2, 3, or 4. The second digit can be chosen from the remaining 5 digits. The third digit can be chosen from the remaining 4 digits.

$$\begin{aligned}\therefore \text{ number of arrangements} &= 3 \times 5 \times 4 \\ &= 60\end{aligned}$$

There are 60 numbers between 200 and 500 with the given digits.

11. The first three digits can only be filled in one way. The fourth digit can be filled in eight ways. The remaining six digits can be filled in ${}_{10}P_6$ ways.

$$\begin{aligned}\therefore \text{ number of telephone numbers} &= 1 \times 8 \times {}_{10}P_6 \\ &= 1 \times 8 \times \frac{10!}{4!} \\ &= 1\,209\,600\end{aligned}$$

A total of 1 209 600 numbers can be formed.

12. The five students may be seated in $5!$ ways.

The remaining fifteen students may be seated in 20 desks in ${}_{20}P_{15}$ ways.

$$\therefore \text{ desk assignments} = 5! \times {}_{20}P_{15}$$
$$= 5! \times \frac{20!}{5!}$$
$$= 20!$$

The number of ways the students can be assigned to the desks is $20!$.

13. a. The first book can be given to any of the seven students in seven ways.

The second book can be given to any of the seven students in seven ways.

$$\therefore \text{ Total} = 7 \times 7$$
$$= 49$$

The books may be given out in 49 ways.

b. The first book can be given to any of the seven students in seven ways.

The second book can be given to any of the remaining six students in six ways.

$$\therefore \text{ Total} = 7 \times 6$$
$$= 42$$

The books may be given out in 42 ways.

14. The marbles and beads may each be arranged in ${}_4P_4$ ways.

The order of marbles and beads may be reversed; thus, multiply by 2.

$$\therefore \text{ number of arrangements} = {}_4P_4 \times {}_4P_4 \times 2$$
$$= 4! \times 4! \times 2$$
$$= 1152$$

The marbles and beads can be arranged in 1152 ways.

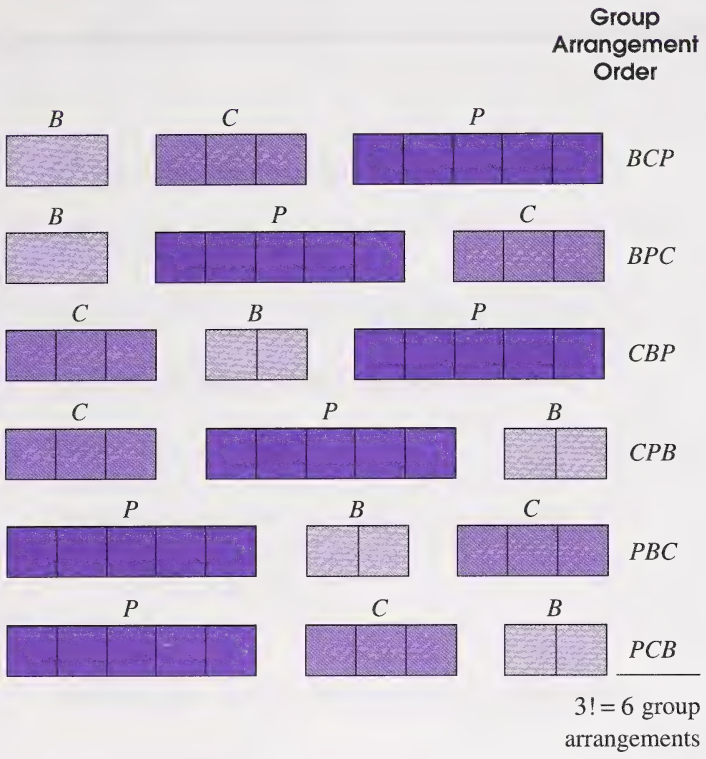
15. The two different biology books can be arranged between themselves in $2!$ ways.

The three different chemistry books can be arranged among themselves in $3!$ ways.

The five different physics books can be arranged among themselves in $5!$ ways.

Each group of books is considered as one unit.

The three different groups of books can be arranged amongst themselves in $3!$ ways. See the diagrams that follow.
B represents the biology books, *C* represents the chemistry books, and *P* represents the physics books.



3! = 6 group arrangements

$\therefore$ number of arrangements $= 2! \times 3! \times 5! \times 3!$

$= 2 \times 6 \times 120 \times 6$

$= 8640$

The books may be arranged in 8640 ways.

16. There are three Es and two Ss in the word PEERLESS.

$$P = \frac{8!}{3!2!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times \cancel{3!}}{\cancel{3!} \times 2 \times 1}$$

$$= 3360$$

Therefore, 3360 permutations can be made.

17. There are eight books altogether.

The three copies of chemistry books are the same.

The five copies of physics books are the same.

$$P = \frac{8!}{3!5!}$$

$$= \frac{8 \times 7 \times 6 \times \cancel{5!}}{3 \times 2 \times 1 \times \cancel{5!}}$$

$$= 56$$

The books can be distributed in 56 ways.

18. There are 11 letters in the word.

There are two Ts, three Es, two Rs, and two Ms.

$$P = \frac{11!}{2!3!2!2!}$$

$$= 831\,600$$

Therefore, 831 600 permutations can be made.

Section 1: Activity 4

1. $(9-1)!=8!$

$=40\,320$

The dancers can form a circle in 40 320 ways.

2. $(12-1)!=11!$

$=39\,916\,800$

The football team can make a huddle in 39 916 800 ways.

3. $(10-1)!=9!$

$=362\,880$

The gardener can plant the flowers in 362 880 ways.

4. $(37-1)!=36!$

There are $36!$ ways to arrange the numbers.

5. $\dfrac{(12-1)!}{2!}=\dfrac{11!}{2!}$

$=19\,958\,400$

The flowers can be arranged in 19 958 400 ways.

6. Assume the girls are seated first. A position is assigned to a girl and the other four girls are arranged in $4!$ ways. The five boys are arranged in $5!$ ways. The total number of circular arrangements is $(5-1)!\times 5!=24\times 120=2880$.

7. The three marbles must be adjacent; thus, consider the three marbles as one group of marbles. These three marbles can be arranged in $3!$ ways. The remaining four marbles plus this one group of marbles gives five marbles. These five marbles can be arranged in $(5-1)!$ ways. The total number of arrangements is $3!\times(5-1)!=6\times 24=144$.

8. $\dfrac{(5-1)!}{2}=\dfrac{4!}{2}$

$=12$

Five keys can be arranged in 12 ways.

9. $\dfrac{(7-1)!}{2}=\dfrac{6!}{2}$

$=360$

The charms can be arranged in 360 ways.

10. $\dfrac{(4-1)!}{2} = \dfrac{3!}{2}$

$= 3$

Three arrangements are possible.

11. The three jewels can be arranged in 3! ways. Consider these three pieces as one.

The twenty-five beads plus this one (jewels) gives twenty-six.

$\therefore$ number of arrangements $= 3! \times \dfrac{(26-1)!}{2}$

$= 3 \times 25!$

A total of $3 \times 25!$ arrangements are possible.

Section 1: Activity 5

1. $P(E) = \dfrac{1}{{}_7P_7}$

$= \dfrac{1}{7!}$

$= \dfrac{1}{5040}$

The probability is $\frac{1}{5040}$.

2. $\dfrac{{}_3P_3}{{}_8P_3} = \dfrac{3!}{\frac{8!}{(8-3)!}}$

$= \dfrac{3!\,5!}{8!}$

$= \dfrac{3 \times 2 \times 1}{8 \times 7 \times 6}$

$= \dfrac{1}{56}$

The probability is $\frac{1}{56}$.

3. There are two black kings, two red queens, and two black jacks.

The number of favourable cases is $2 \times 2 \times 2$.

The number of possible cases is ${}_{52}P_3$

$\therefore$ probability $= \dfrac{2 \times 2 \times 2}{{}_{52}P_3}$

$= \dfrac{8}{52 \times 51 \times 50}$

$= \dfrac{1}{16\,575}$

The probability is $\frac{1}{16\,575}$.

4. a. The probability that all three balls are red is

$$P(E) = \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}$$
$$= 0.027$$

Two red balls and one blue ball can be arranged

$\frac{{}_3P_3}{2!} = 3$ ways (red, red, blue; red, blue, red; blue, red, red)

The probability of two red balls and one blue ball is as follows:

$$\left(\frac{3}{10} \times \frac{3}{10} \times \frac{7}{10}\right) \times \frac{{}_3P_3}{2!} = 0.063 \times 3$$
$$= 0.189$$

$$\therefore \text{ Total probability} = 0.027 + 0.189$$
$$= 0.216$$

The total probability is 0.216.

b. The probability of all three red balls is as follows:

$$\frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} \doteq 0.008\,333\,333$$

The probability of two red balls and one blue ball is as follows:

$$\left(\frac{3}{10} \times \frac{2}{9} \times \frac{7}{8}\right) \times \frac{{}_3P_3}{2!} = (0.058\,333\,333) \times 3$$
$$= 0.175$$

$$\therefore \text{ Total Probability} \doteq 0.008\,333\,333 + 0.175$$
$$\doteq 0.1833$$

The probability is about 0.1833.

Section 1: Follow-up Activities

Extra Help

1. A president may be chosen from any of the ten people. Once a president is chosen, a secretary can be chosen from the remaining 9 people. Finally, the treasurer can be chosen from the remaining 8 people.

10	9	8

$10 \times 9 \times 8 = 720$

There are 720 ways to choose the three positions.

2. $(n+r-4)! = (n+r-4)(n+r-5)(n+r-6)\ldots 2 \times 1$

3. Examine the factors. The factors are reducing by 1. The first factor (leading factor) is $(p-q+6)$. Therefore, the factorial form is $(p-q+6)!$.

4. Any four letters $= {}_4P_4$ Any three letters $= {}_4P_3$
Any two letters $= {}_4P_2$ Any one letter $= {}_4P_1$

$$\begin{aligned}\text{Total} &= {}_4P_4 + {}_4P_3 + {}_4P_2 + {}_4P_1 \\ &= 4! + 4! + 12 + 4 \\ &= 24 + 24 + 12 + 4 \\ &= 64\end{aligned}$$

Therefore, 64 words can be formed.

5. Two letters are chosen from three letters. This can be done in ${}_3P_2$ ways. Two-digit numbers are formed from four digits. These can be chosen in ${}_4P_2$ ways.

$$\begin{aligned}\therefore \text{ Total number of arrangements} &= {}_3P_2 \times {}_4P_2 \\ &= \frac{3!}{1!} \times \frac{4!}{2!} \\ &= 6 \times 12 \\ &= 72\end{aligned}$$

There can be 72 licence plates made.

6. The girls can be arranged in $3!$ ways among themselves.
The boys can be arranged in $2!$ ways.
There are a total of three positions where the girls may be seated (front of the row, middle of the row, or back of the row.)

$$3! \times 2! \times 3 = 36$$

The group can be seated in 36 ways.

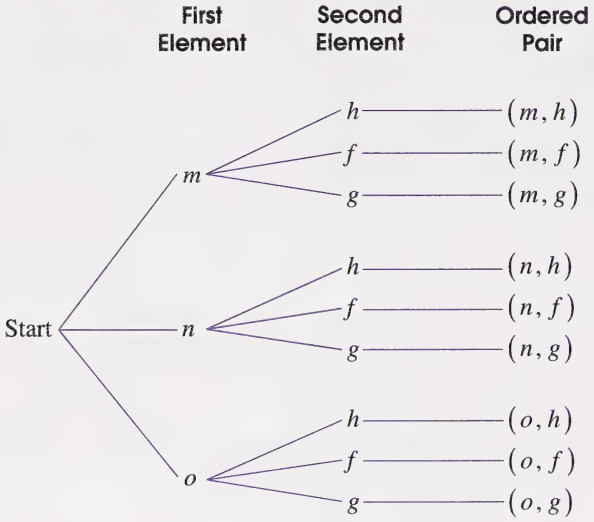
7. The math books can be arranged in $4!$ ways.
The physics books can be arranged in $6!$ ways.
There are three possible positions where the math books can be placed.

$$4! \times 6! \times 3 = 51\,840$$

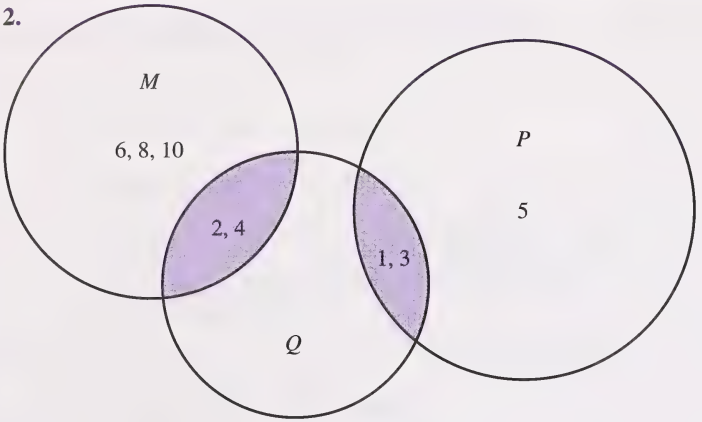
The books can be arranged in 51 840 ways.

Enrichment

1.



2.



$n(P \cap Q) = 2$

$n(M \cap P) = 0$

$n(M \cup P) = 8$

$n(P \cup Q) = 5$

Section 2: Activity 1

1. $_{17}C_3 = \dfrac{17!}{(17-3)!3!}$

$= \dfrac{17!}{14!3!}$

$= \dfrac{17 \times 16 \times 15}{3 \times 2 \times 1}$

$= 680$

2. $_4C_3 + {}_5C_0 - {}_6C_1 = \dfrac{4!}{3!1!} + \dfrac{5!}{0!5!} - \dfrac{6!}{5!1!}$

$= 4 + 1 - 6$

$= -1$

3.

LS	RS
$_6C_3 + {}_6C_4$	$_7C_4$
$= \dfrac{6!}{3!3!} + \dfrac{6!}{4!2!}$	$= \dfrac{7!}{4!3!}$
$= \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1} + \dfrac{6 \times 5}{2 \times 1}$	$= \dfrac{7 \times 6 \times 5}{3 \times 2}$
$= 20 + 15$	$= 35$
$= 35$	

LS = RS

4. $_9C_2 = \dfrac{9!}{7!2!}$

$= \dfrac{9 \times 8}{2 \times 1}$

$= 36$

There will be 36 games played.

5. $${}_{12}C_8 = \frac{12!}{4!8!}$$

$$= \frac{12\times11\times10\times9}{4\times3\times2\times1}$$

$$= 495$$

The vote could have resulted 495 different ways.

6. $${}_{15}C_6 = \frac{15!}{9!\times6!}$$

$$= \frac{15\times14\times13\times12\times11\times10}{6\times5\times4\times3\times2\times1}$$

$$= 5005$$

The committee could have been formed 5005 different ways.

7. $${}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10} = \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$$

$$= \frac{10\times9\times8}{3\times2\times1} + \frac{10\times9}{2\times1} + 10 + 1$$

$$= 120 + 45 + 10 + 1$$

$$= 176$$

The pictures may be selected in 176 ways.

8. $${}_{12}C_3 + {}_{12}C_2 + {}_{12}C_1 = \frac{12!}{9!3!} + \frac{12!}{10!2!} + \frac{12!}{11!1!}$$

$$= 220 + 66 + 12$$

$$= 298$$

Priscilla can choose the cookies in 298 ways.

9. $${}_8C_3 + {}_8C_4 + {}_8C_5 + {}_8C_6 + {}_8C_7 + {}_8C_8$$

$$= \frac{8!}{5!3!} + \frac{8!}{4!4!} + \frac{8!}{3!5!} + \frac{8!}{2!6!} + \frac{8!}{1!7!} + \frac{8!}{0!8!}$$

$$= 56 + 70 + 56 + 28 + 8 + 1$$

$$= 219$$

The students may be chosen in 219 ways.

10. $$\frac{{}_{10}C_5}{2} = \frac{\frac{10!}{5!5!}}{2}$$

$$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times2}$$

$$= 126$$

The players can be separated in 126 ways.

11. $\frac{{}_8C_4}{2} = \frac{\frac{8!}{4!4!}}{2}$

$= \frac{8\times7\times6\times5}{4\times3\times2\times1}\times\frac{1}{2}$

$= 35$

The pups may be separated in 35 ways.

12. ${}_3C_1 \times {}_8C_2 \times {}_{10}C_3 \times {}_9C_4 = \frac{3!}{2!1!}\times\frac{8!}{6!2!}\times\frac{10!}{7!3!}\times\frac{9!}{5!4!}$

$= 3\times28\times120\times126$

$= 1\,270\,080$

Krish can buy the groceries in 1 270 080 ways.

13. ${}_{21}C_9 \times {}_{12}C_7 \times {}_5C_5 = \frac{21!}{9!12!}\times\frac{12!}{7!5!}\times1$

$= 293\,930\times792\times1$

$= 232\,792\,560$

The class can be divided in 232 792 560 ways.

14. $2^{10} - 1 = 1023$

Helena can choose the candies in 1023 ways.

15. $2^7 - 1 = 127$

Ralph can make a choice in 127 ways.

16. $2^8 = 256$

The candidates can be elected in 256 ways.

17. **a.** ${}_{40}C_5 = 658\,008$

A total of 658 008 hands can be formed.

b. ${}_{12}C_2 \times {}_{40}C_3 = \frac{12!}{10!2!}\times\frac{40!}{37!3!}$

$= \frac{12\times11}{1\times2}\times\frac{40\times39\times38}{3\times2\times1}$

$= 652\,080$

A total of 652 080 hands can be formed.

c. The kings are always excluded. There are 48 cards left. The two red queens are always included. The other three cards must be selected from the $(48-2)=46$ cards left.

${}_2C_2 \times {}_{46}C_3 = 1\times\frac{46!}{43!3!}$

$= 15\,180$

A total of 15 180 hands can be formed.

18. The number of any five-card hands $= {}_{52}C_5$

$= 2\,598\,960$

No king included $= {}_{48}C_5$

$= 1\,712\,304$

At least one king included $= 2\,598\,960 - 1\,712\,304$

$= 886\,656$

A total of 886 656 hands can be made.

19. One drink: ${}_5C_1 = \dfrac{5!}{4!1!}$

$= 5$

No sandwich (choose four desserts): ${}_6C_4 = \dfrac{6!}{2!4!}$

$= 15$

Any four items (sandwich plus dessert): ${}_9C_4 = \dfrac{9!}{5!4!}$

$= 126$

$\therefore$ number of packages $= {}_5C_1\left({}_9C_4 - {}_6C_4\right)$

$= 5(126 - 15)$

$= 555$

A total of 555 lunch packages is possible.

Alternate Method

One drink, one sandwich, and three desserts

$= {}_5C_1 \times {}_3C_1 \times {}_6C_3$

$= 5 \times 3 \times 20$

$= 300$

One drink, two sandwiches, and two desserts

$= {}_5C_1 \times {}_3C_2 \times {}_6C_2$

$= 5 \times 3 \times 15$

$= 225$

One drink, three sandwiches, and one dessert

$= 5 \times 1 \times 6$

$= 30$

Number of packages $= 300 + 225 + 30$

$= 555$

A total of 555 lunch packages is possible.

20. The number of ways to choose at least one shirt is $2^5 - 1$.
The number of ways to choose at least one pair of pants is $2^6 - 1$.
The number of ways to choose any number of hats is 2^3.
The number of ways to choose the three items is as follows:

$$\begin{aligned}(2^5 - 1)(2^6 - 1)(2^3) &= (31)(63)(8) \\ &= 15\,624\end{aligned}$$

Tui can make 15 624 selections.

21. The number of straight lines formed by joining any two of twelve points is ${}_{12}C_2 = 66$.

The number of straight lines formed by joining any two of four points is ${}_4C_2 = 6$.

Since four of the twelve points form one straight line, the number of straight lines formed by joining these twelve points is $66 - 6 + 1 = 61$.

22.
$$\begin{aligned}{}_{14}C_2 - 14 &= \frac{14!}{12!\,2!} - 14 \\ &= 91 - 14 \\ &= 77\end{aligned}$$

A fourteen-sided polygon has 77 diagonals.

23.
$$\begin{aligned}{}_5C_2 \times {}_6C_2 &= 10 \times 15 \\ &= 150\end{aligned}$$

A total of 150 parallelograms can be formed.

Section 2: Activity 2

1. The number of ways to choose any two cards is ${}_9C_2$.
The number of ways to choose any two even numbers is ${}_4C_2$.

$$\begin{aligned}P(E) &= \frac{{}_4C_2}{{}_9C_2} \\ &= \frac{\frac{4!}{2!\,2!}}{\frac{9!}{7!\,2!}} \\ &= \frac{6}{36} \\ &= \frac{1}{6}\end{aligned}$$

The probability that both cards are even numbers is $\frac{1}{6}$.

2. Jody can be chosen in ${}_1C_1$ ways.

The remaining students are chosen in ${}_{24}C_4$ ways.

The number of ways to choose any five students is ${}_{25}C_5$.

$$P(E) = \frac{{}_1C_1 \times {}_{24}C_4}{{}_{25}C_5}$$
$$= \frac{\frac{1!}{0!1!} \times \frac{24!}{20!4!}}{\frac{25!}{20!5!}}$$
$$= \frac{1 \times 10\,626}{53\,130}$$
$$= \frac{1}{5}$$

The probability that Jody will be chosen is $\frac{1}{5}$.

3. Find the number of ways to choose exactly one winning ticket.

$${}_2C_1 \times {}_{48}C_1 = \frac{2!}{1!1!} \times \frac{48!}{47!1!}$$
$$= 96$$

Find the number of ways to choose any two tickets.

$${}_{50}C_2 = \frac{50!}{48!2!}$$
$$= 1225$$

$$P(E) = \frac{96}{1225}$$

The probability that Horst has one winning ticket is $\frac{96}{1225}$.

4. Find the number of ways in which two cards can be selected out of 52 cards.

$${}_{52}C_2 = \frac{52!}{50!2!}$$
$$= 1326$$

Find the number of ways to select two cards out of twenty-six red cards.

$${}_{26}C_2 = \frac{26!}{24!2!}$$
$$= 325$$

$$\therefore P(E) = \frac{325}{1326}$$

The probability that both cards are red is $\frac{325}{1326}$.

5. Find the number of ways of drawing one marble.

$${}_{12}C_1 = \frac{12!}{11!1!}$$
$$= 12$$

Find the number of ways of drawing one red marble.

$${}_4C_1 = \frac{4!}{3!1!}$$
$$= 4$$

Find the number of ways of drawing one green marble.

$$
\begin{aligned}
{}_2C_1 &= \frac{2!}{1!1!} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\therefore P(E) &= \frac{4+2}{12} \\
&= \frac{1}{2}
\end{aligned}
$$

The probability of drawing a red or green marble is $\frac{1}{2}$.

6. Only four members can be selected in ${}_{11}C_4$ ways.
The two Alberta delegates can be selected in ${}_4C_2$ ways.
The one N.W.T. delegate can be selected in ${}_3C_1$ ways.
The one Newfoundland delegate can be selected in ${}_3C_1$ ways.

$$
\begin{aligned}
\therefore P(E) &= \frac{{}_4C_2 \times {}_3C_1 \times {}_3C_1}{{}_{11}C_4} \\
&= \frac{\frac{4!}{2!2!} \times \frac{3!}{2!1!} \times \frac{3!}{2!1!}}{\frac{11!}{7!4!}} \\
&= \frac{6 \times 3 \times 3}{330} \\
&= \frac{9}{55}
\end{aligned}
$$

The probability of choosing these delegates is $\frac{9}{55}$.

7. number of favourable cases $= 1 \times 3$
$= 3$

number of possible cases $= {}_2C_1 \times {}_6C_1$
$= 12$

$$
\begin{aligned}
\therefore P(E) &= \frac{3}{12} \\
&= \frac{1}{4}
\end{aligned}
$$

The probability is $\frac{1}{4}$.

8. The number of ways of choosing a committee of four is ${}_{25}C_4$.

The number of ways that a committee of four boys can be chosen is ${}_{15}C_4$.

$$
\begin{aligned}
\therefore P(E) &= 1 - \frac{{}_{15}C_4}{{}_{25}C_4} \\
&= 1 - \frac{\frac{15!}{11!4!}}{\frac{25!}{21!4!}} \\
&= 1 - \frac{1365}{12\,650} \\
&= \frac{11\,285}{12\,650} \\
&= \frac{2257}{2530}
\end{aligned}
$$

The probability that at least one girl is chosen is $\frac{2257}{2530}$.

Section 2: Activity 3

1. a. $(x+2y)^8 = {}_8C_0x^8 + {}_8C_1x^7(2y) + {}_8C_2x^6(2y)^2 + {}_8C_3x^5(2y)^3 + {}_8C_4x^4(2y)^4 + {}_8C_5x^3(2y)^5 + {}_8C_6x^2(2y)^6$
$+ {}_8C_7x(2y)^7 + {}_8C_8(2y)^8$
$= x^8 + 8x^7(2y) + 28x^6(2y)^2 + 56x^5(2y)^3 + 70x^4(2y)^4 + 56x^3(2y)^5 + 28x^2(2y)^6 + 8x(2y)^7 + (2y)^8$
$= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + 1120x^4y^4 + 1792x^3y^5 + 1792x^2y^6 + 1024xy^7 + 256y^8$

b. $(2x-y)^7 = {}_7C_0(2x)^7 + {}_7C_1(2x)^6(-y) + {}_7C_2(2x)^5(-y)^2 + {}_7C_3(2x)^4(-y)^3 + {}_7C_4(2x)^3(-y)^4$
$+ {}_7C_5(2x)^2(-y)^5 + {}_7C_6(2x)(-y)^6 + {}_7C_7(-y)^7$
$= 128x^7 + 7(64x^6)(-y) + (21)(32x^5)(y^2) + (35)(16x^4)(-y^3) + (35)(8x^3)(y^4)$
$+ 21(4x^2)(-y^5) + 7(2x)(y^6) - y^7$
$= 128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

c. $\left(x^2 - \frac{1}{2}y\right)^5 = {}_5C_0\left(x^2\right)^5 + {}_5C_1\left(x^2\right)^4\left(-\frac{1}{2}y\right) + {}_5C_2\left(x^2\right)^3\left(-\frac{1}{2}y\right)^2 + {}_5C_3\left(x^2\right)^2\left(-\frac{1}{2}y\right)^3$
$+ {}_5C_4\left(x^2\right)\left(-\frac{1}{2}y\right)^4 + {}_5C_5\left(-\frac{1}{2}y\right)^5$
$= x^{10} + (5)\left(x^8\right)\left(-\frac{1}{2}y\right) + (10)\left(x^6\right)\left(-\frac{1}{2}y\right)^2 + (10)\left(x^4\right)\left(-\frac{1}{2}y\right)^3 + (5)\left(x^2\right)\left(-\frac{1}{2}y\right)^4 + \left(-\frac{1}{2}y\right)^5$
$= x^{10} - \frac{5}{2}x^8y + \frac{5}{2}x^6y^2 - \frac{5}{4}x^4y^3 + \frac{5}{16}x^2y^4 - \frac{1}{32}y^5$

2. $(a+b-c)^4 = [a+(b-c)]^4$

$= {}_4C_0 a^4 + {}_4C_1 a^3 (b-c) + {}_4C_2 a^2 (b-c)^2 + {}_4C_3 a(b-c)^3 + {}_4C_4 (b-c)^4$

$= a^4 + 4a^3(b-c) + 6a^2(b-c)^2 + 4a(b-c)^3 + (b-c)^4$

$= a^4 + 4a^3 b - 4a^3 c + 6a^2\left(b^2 - 2bc + c^2\right) + 4a\left(b^3 - 3b^2 c + 3bc^2 - c^3\right) + \left(b^4 - 4b^3 c + 6b^2 c^2 - 4bc^3 + c^4\right)$

$= a^4 + 4a^3 b - 4a^3 c + 6a^2 b^2 - 12a^2 bc + 6a^2 c^2 + 4ab^3 - 12ab^2 c + 12abc^2 - 4ac^3 + b^4 - 4b^3 c + 6b^2 c^2$

$\quad - 4bc^3 + c^4$

3. $2^7 = (1+1)^7$

$= {}_7C_0 (1)^7 + {}_7C_1 (1)^6 (1) + {}_7C_2 (1)^5 (1)^2 + {}_7C_3 (1)^4 (1)^3 + {}_7C_4 (1)^3 (1)^4 + {}_7C_5 (1)^2 (1)^5 + {}_7C_6 (1)(1)^6 + {}_7C_7 (1)^7$

$= {}_7C_0 + {}_7C_1 + {}_7C_2 + {}_7C_3 + {}_7C_4 + {}_7C_5 + {}_7C_6 + {}_7C_7$

4. $t_{k+1} = {}_nC_k x^{n-k} y^k$

$t_9 = t_{8+1} = {}_{15}C_8 x^{15-8} (-2)^8$

$= \frac{15!}{7!8!} x^7 (-2)^8$

$= 6435 x^7 (256)$

$= 1\,647\,360 x^7$

5. The middle term is the fourth term since there are seven terms in the expansion of $\left(x^2 - \sqrt{y}\right)^6$.

$$t_{k+1} = {}_nC_k\, x^{n-k} y^k$$

$$t_4 = t_{3+1} = {}_6C_3 \left(x^2\right)^{6-3} \left(-\sqrt{y}\right)^3$$

$$= \frac{6!}{3!3!}\left(x^2\right)^3 \left(-\sqrt{y}\right)^3$$

$$= -20x^6 y^{\frac{3}{2}}$$

6. The exponent of the second term is 8. Thus, $k = 8$.

$$\therefore\ {}_nC_k = {}_{16}C_8$$

7. In the expansion $(x+y)^n$, the sum of the exponents for any one term in the expansion is always equal to the value n.

$$x + 7 = 12$$
$$x = 5$$

8. ${}_nC_k\, x^{n-k} y^k = \frac{15}{16} x^4 y^2$

$$k = 2$$
$$k + 1 = 2 + 1$$
$$= 3$$

Therefore, $\frac{15}{16} x^4 y^2$ is the third term.

9. ${}_{14}C_6 = \frac{14!}{8!6!}$

$$= {}_{14}C_8$$

10. **a.** $x = 15 - 10$

$$= 5$$

b. $x = 15 - 8$

$$= 7$$

11. $n = 4$: 1 4 6 4 1

$n = 5$: 1 5 10 10 5 1

$n = 6$: 1 6 15 20 15 6 1

$n = 7$: 1 7 21 35 35 21 7 1

$$(x+y)^7 = x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4$$
$$+ 21x^2 y^5 + 7xy^6 + y^7$$

Section 2: Follow-up Activities

Extra Help

1. Three programs are selected in ${}_5C_3$ ways.
Four programs are selected in ${}_5C_4$ ways.
Five programs are selected in ${}_5C_5$ ways.

sum of the different selections $= {}_5C_3 + {}_5C_4 + {}_5C_5$

$$= \frac{5!}{2!3!} + \frac{5!}{1!4!} + \frac{5!}{0!5!}$$
$$= 10 + 5 + 1$$
$$= 16$$

The programs may be selected in 16 ways.

2. Two shirts can be selected from five shirts in ${}_5C_2$ ways.
Three ties can be selected from six ties in ${}_6C_3$ ways.

$${}_5C_2 \times {}_6C_3 = \frac{5!}{3!2!} \times \frac{6!}{3!3!}$$
$$= 10(20)$$
$$= 200$$

The selection can be made in 200 ways.

3. You can choose 0 to 4 pens in 2^4 ways.
Since 0 is not included, the selection is $2^4 - 1$.

$$\therefore\ 2^4 - 1 = 16 - 1$$
$$= 15$$

You can make a choice in 15 ways.

4. The number of ways to choose no beverage but five main dishes from eleven main dishes $= {}_{11}C_5$

$$= \frac{11!}{5!6!}$$
$$= 462$$

There are sixteen items altogether.

Any five items $= {}_{16}C_5$

$$= \frac{16!}{5!11!}$$
$$= 4368$$

Find the number of ways five items can be chosen with at least one beverage.

$${}_{16}C_5 - {}_{11}C_5 = 4368 - 462$$
$$= 3906$$

The items can be chosen in 3906 ways.

Enrichment

1. The number of ways to form a committee is ${}_{26}C_6$.

The number of ways to choose a chairperson and a secretary is ${}_6P_2$.

$$
\begin{aligned}
{}_{26}C_6 \times {}_6P_2 &= \frac{26!}{6!\,20!} \times \frac{6!}{4!} \\
&= \frac{26\times25\times24\times23\times22\times21}{6\times5\times4\times3\times2\times1} \times \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1} \\
&= 6\,906\,900
\end{aligned}
$$

There are 6 906 900 ways to designate the chairperson and secretary.

2. To choose ten different charms, there are ${}_{15}C_{10}$ ways.

Find the number of arrangements with ten charms.

$$\frac{(10-1)!}{2} = \frac{9!}{2}$$

$$
\begin{aligned}
\therefore \text{ number of arrangements} &= {}_{15}C_{10} \times \frac{9!}{2} \\
&= \frac{15!}{5!\,10!} \times \frac{9!}{2} \\
&= \frac{15\times14\times13\times12\times11}{1} \times \frac{9\times8\times7\times6}{2} \\
&= 544\,864\,320
\end{aligned}
$$

Linda could make 544 864 320 arrangements.

3. The two physics books can be arranged in ${}_2P_2 = 2!$ ways.

The three novels can be selected in ${}_3C_2$ ways.

The four books can be arranged in ${}_3P_3$ ways. The two physics books are considered as one book since they must be together.

$$
\begin{aligned}
\therefore \text{ number of ways} &= {}_2P_2 \times {}_3C_2 \times {}_3P_3 \\
&= 2! \times \frac{3!}{2!} \times 3! \\
&= 36
\end{aligned}
$$

The total number of ways to select four books and arrange them on a bookshelf is 36.

Section 3: Activity 1

1. $x\sigma_n = 4.1$

2. The standard deviation for Crunchies is 5.7. The standard deviation for Munchies is 3.3.

The Munchies company is more consistent in its packaging.

Section 3: Activity 2

1.

$25 = \mu + 2\sigma$ ①

$10 = \mu - 1\sigma$ ②

① − ②: $15 = 3\sigma$

$5 = \sigma$

$10 = \mu - 1\sigma$

$10 = \mu - 1(5)$

$10 = \mu - 5$

$15 = \mu$

$a = \mu - 3\sigma$

$= 15 - 3(5)$

$= 0$

$b = \mu - 2\sigma$

$= 15 - 2(5)$

$= 5$

$c = \mu$

$= 15$

$d = \mu + 1\sigma$

$= 15 + 1(5)$

$= 20$

$e = \mu + 3\sigma$

$= 15 + 3(5)$

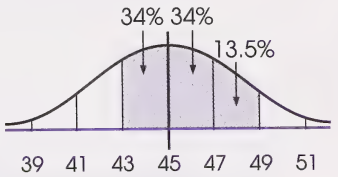
$= 30$

2. **a.** Since 2 g is one standard deviation, the question asks how many of the chocolates are within one standard deviation of the mean. 68% of the 10 000 chocolates will be within one standard deviation of the mean.

$$68\% \times 10\,000 = 0.68 \times 10\,000$$
$$= 6800$$

There will be 6800 chocolates within 2 g of the required mass.

b. First, construct a graph of the situation.

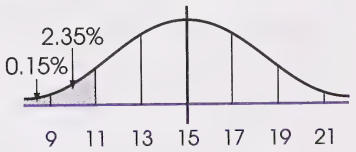


The percent rejected will be $100\% - (34\% + 34\% + 13.5\%)$ or 18.5%.

$$18.5\% \times 100\,000 = 0.185 \times 100\,000$$
$$= 18\,500$$

There will be 18 500 chocolates rejected.

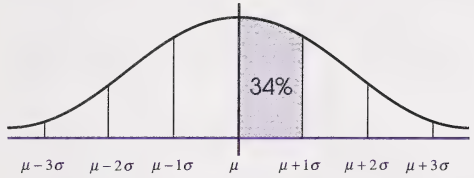
3. First, construct a graph using this information.



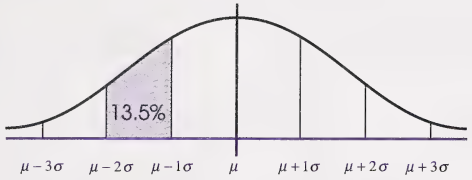
2.5% is below two standard deviations less than the mean. Two standard deviations less than the mean is eleven years.

The company will offer a guarantee of eleven years.

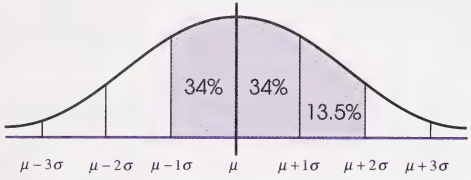
4. a. The area between μ and $\mu+\sigma$ is 34%.



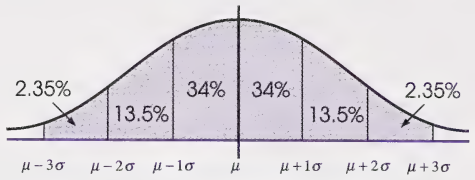
b. The area between $\mu-2\sigma$ and $\mu-\sigma$ is 13.5%.



c. The area between $\mu+2\sigma$ and $\mu-\sigma$ is 81.5%.



d. The area between $\mu-3\sigma$ and $\mu+3\sigma$ is 99.7%.



$$2.35\% + 13.5\% + 34\% + 34\% + 13.5\% + 2.35\% = 99.7\%$$

Section 3: Activity 3

1. The z-score is -2.

2. The mean is 197.

The standard deviation is approximately 5.6.

The z-scores are as follows:

$$z = \frac{x-\mu}{\sigma} \qquad z = \frac{x-\mu}{\sigma}$$
$$= \frac{188-197}{5.6} \qquad = \frac{190-197}{5.6}$$
$$= \frac{-9}{5.6} \qquad = \frac{-7}{5.6}$$
$$\doteq -1.61 \qquad = -1.25$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{192-197}{5.6}$$
$$= \frac{-5}{5.6}$$
$$\doteq -0.89$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{196-197}{5.6}$$
$$= \frac{-1}{5.6}$$
$$\doteq -0.18$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{198-197}{5.6}$$
$$= \frac{1}{5.6}$$
$$\doteq 0.18$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{199-197}{5.6}$$
$$= \frac{2}{5.6}$$
$$\doteq 0.36$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{202-197}{5.6}$$
$$= \frac{5}{5.6}$$
$$\doteq 0.89$$

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{207-197}{5.6}$$
$$= \frac{10}{5.6}$$
$$\doteq 1.79$$

3. The z-score for 185 hits is as follows:

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{185-150}{20}$$
$$= \frac{35}{20}$$
$$= 1.75$$

The area for the z-score 1.75 is 0.4599.

The following represents the area to the right of the z-score 1.75.

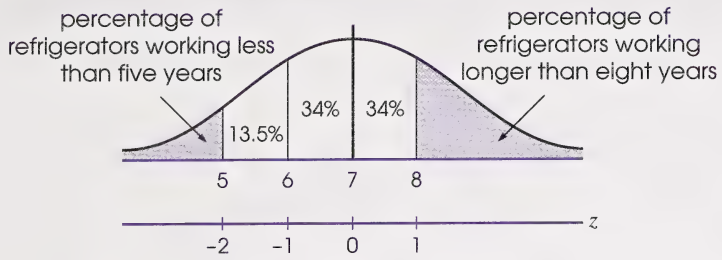
$$0.5 - 0.4599 = 0.0401$$

The following represents the number of seasons the player had at least 185 hits.

$$0.0401 \times 25 = 1.0025$$
$$= 1$$

The baseball player had at least 185 hits in only one season.

4.



Method 1: Using Approximate Area

The percentage of refrigerators working longer than eight years is $50\% - 34\% = 16\%$.

$$\frac{16}{100} \times 1200 = 192$$

Therefore, 192 refrigerators will work longer than eight years.

The percentage of refrigerators working less than five years is $50\% - (34\% + 13.5\%) = 2.5\%$.

$$\frac{2.5}{100} \times 1200 = 30$$

Therefore, 30 refrigerators must be replaced.

Method 2: Using z-Scores

To determine the number of refrigerators working longer than eight years, the area to the right of eight years must be calculated.

$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{8 - 7}{1}$$
$$= 1 \quad \text{(This matches the diagram.)}$$

From the table, when $z = 1$, the area is 0.3413. This is the area to the right of the mean between $z = 0$ and $z = 1$. The required area to the right of eight years is $0.5000 - 0.3413 = 0.1587$.

$$0.1587 \times 1200 = 190.44$$

Thus, about 190 refrigerators will work longer than eight years.

The area to the left of five years must be calculated.

$$z = \frac{5 - 7}{1}$$
$$= -2 \quad \text{(This matches the diagram.)}$$

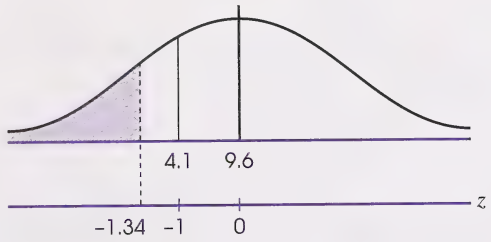
From the table, when $z = -2$, the area is 0.4772. This is the area to the left of the mean between $z = 0$ and $z = -2$. The required area to the left of five years is $0.5000 - 0.4772 = 0.0228$.

$$0.0228 \times 1200 = 27.36$$

Thus, about 27 refrigerators must be replaced.

Note: The answers vary since Method 1 uses the approximate percentages for the areas. Using z-scores gives a more accurate solution.

5. Since the company will only replace 9% of the toasters, then $50\% - 9\% = 41\%$ is the area to the left of the mean. From the z-score table an area of 41% to the left of the mean gives a z-score value of -1.34. The value of x is determined by using the z-score formula.



$$z = \frac{x - \mu}{\sigma}$$

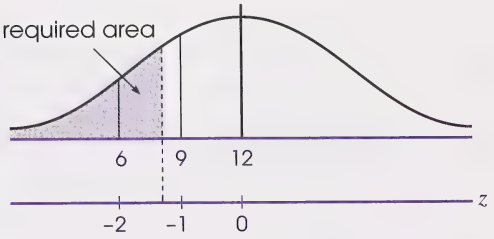
$$-1.34 = \frac{x - 9.6}{5.5}$$

$$-7.37 = x - 9.6$$

$$x = 2.23$$

The guarantee period for the toasters should be 2.2 years.

6. Determine the z-score for eight years by using the formula.



$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{8 - 12}{3}$$

$$= -1.33\dot{3}$$

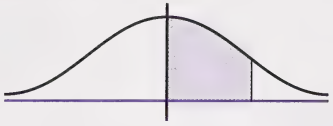
From the table, the area for this z-score is 0.4082. This is the area from the mean to $z = -1.33\dot{3}$. The required area for the period less than eight years is $0.5000 - 0.4082 = 0.0918$.

Thus, 9% of the compressors have to be repaired under this eight-year warranty period.

Section 3: Activity 4

1. a. $P(z \le 0) = 0.5$ or 50%

b. $P(0 \le z \le 2) = 0.4772$ or 47.72%

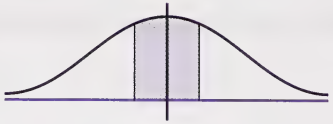


c. The area for a z-score of 1.80 is 0.4641.

$P(z \ge 1.80) = 0.5 - 0.4641$

$= 0.0359$ or 3.59%

d.



The area for a z-score of -0.89 is 0.3133.

The area for a z-score of 1.24 is 0.3925.

$P(-0.89 \le z \le 1.24) = 0.3133 + 0.3925$

$= 0.7058$ or 70.58%

e.

The area for the z-score -2.37 is 0.4911.

The area for the z-score -0.92 is 0.3212.

$P(-2.37 \le z \le -0.92) = 0.4911 - 0.3212$

$= 0.1699$ or 16.99%

2. a. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{15-13}{1.3}$

$\doteq 1.54$

The area for the z-score 1.54 is 0.4382.

$P(\geq 15 \text{ years}) = 0.5 - 0.4382$

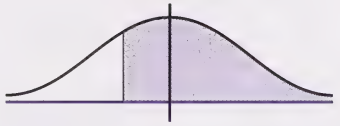
$= 0.0618$ or 6.18%

The probability that a cat will live longer than fifteen years is 0.0618 or 6.18%.

b. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{12-13}{1.3}$

$\doteq -0.77$



The area for the z-score -0.77 is 0.2794.

$P(\geq 12 \text{ years}) = 0.5 + 0.2794$

$= 0.7794$ or 77.94%

The probability that a cat will live longer than twelve years is 0.7794 or 77.94%.

c. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{10-13}{1.3}$

$\doteq -2.31$

The area for the z-score -2.31 is 0.4896.

$P(\leq 10 \text{ years}) = 0.5 - 0.4896$

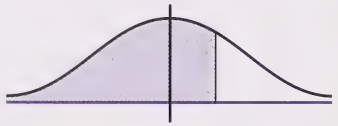
$= 0.0104$ or 1.04%

The probability that a cat will live less than ten years is 0.0104 or 1.04%.

d. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{14-13}{1.3}$

$\doteq 0.77$



The area for the z-score 0.77 is 0.2794.

$P(\leq 14 \text{ years}) = 0.5 + 0.2794$

$= 0.7794$ or 77.94%

The probability that a cat will live less than fourteen years is 0.7794 or 77.94%.

3. a. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{32-35}{6.5}$

$= -0.46$

The area for the z-score -0.46 is 0.1772.

$P(\leq 32 hours) = 0.5 - 0.1772$

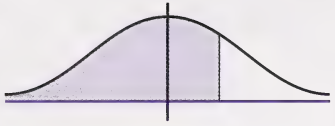
$= 0.3228$ or 32.28%

The probability that a battery will last less than 32 h is 0.3228 or 32.28%.

b. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{43-35}{6.5}$

$\doteq 1.23$



The area for the z-score 1.23 is 0.3907.

$P(\leq 43 \text{ hours}) = 0.5 + 0.3907$

$= 0.8907$ or 89.07%

The probability that a battery will last less than 43 h is 0.8907 or 89.07%.

c. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{47-35}{6.5}$

$\doteq 1.85$

The area for the z-score 1.85 is 0.4678.

$P(\geq 47 \text{ hours}) = 0.5 - 0.4678$

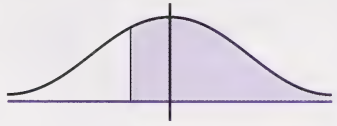
$= 0.0322$ or 3.22%

The probability that a battery will last more than 47 h is 0.0322 or 3.22%.

d. $z = \dfrac{x-\mu}{\sigma}$

$= \dfrac{33-35}{6.5}$

$\doteq -0.31$



The area for the z-score -0.31 is 0.1217.

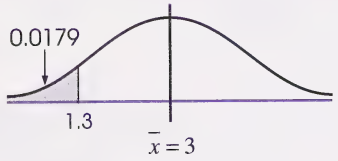
$P(\geq 33 \text{ hours}) = 0.5 + 0.1217$

$= 0.6217$ or 62.17%

The probability that a battery will last more than 33 h is 0.6217 or 62.17%.

4. Find the area close to the mean.

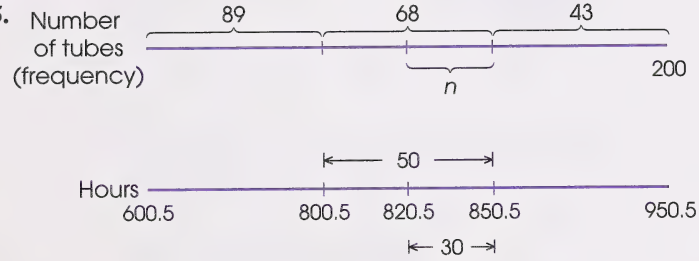
$$\text{Area} = 0.5000 - 0.0179$$
$$= 0.4821$$



$$z = \frac{x - \mu}{\sigma}$$
$$-2.1 = \frac{1.3 - 3.0}{\sigma}$$
$$\sigma = \frac{-1.7}{-2.1}$$
$$\doteq 0.81$$

The standard deviation for the shelf period is about 0.81 d.

5.



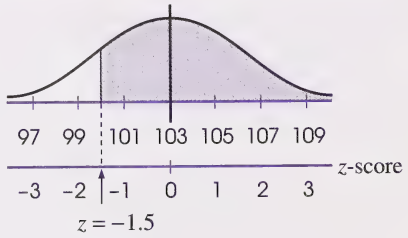
$$\frac{50}{68} = \frac{30}{n}$$
$$n = \frac{30 \times 68}{50}$$
$$= 40.8$$

The number of tubes lasting longer than 820.5 h is $40.8 + 27 + 16 = 83.8$.

$$P = \frac{\text{number of tubes that last longer than 820.5 h}}{\text{total number of tubes}}$$
$$= \frac{83.8}{200}$$
$$= 0.419$$
$$= 0.42$$

The probability a tube will last longer than 820.5 h is 0.42.

6.
$$z = \frac{x - \mu}{\sigma}$$
$$= \frac{100 - 103}{2}$$
$$= \frac{-3}{2}$$
$$= -1.5$$



The area for $z = -1.5$ is 0.4332.

The probability that a box will contain 100 or more tacks is $0.5 + 0.4332 = 0.9332$.

Section 3: Activity 5

1. Possible suggestions for preparing yourself for the diploma exam are as follows:

- Prepare a course review schedule—you should be preparing at least two weeks before the exam.
- Obtain and review exam schedules, rules, and policies. The schedules will be supplied at the appropriate time by your school, learning facilitator, or teacher.
- Identify and collect examples of questions and directing words—question types were examined briefly in the introduction to Module 1 and will be explored in greater detail later in this activity. The directing words are given in the Appendix of Module 1.
- Make summaries and point-form outlines—this step is crucial to your tying together of all the concepts studied in this course. It will help increase your overall understanding.
- Use memory aids—several memory aids have been given throughout the course; develop some of your own as well. It is amazing how memory aids help you focus on the important concepts.

2. The following are possible strategies for answering multiple-choice questions:

- Answer all the questions. You are not penalized for guessing.
- Answer the question first without looking at the alternatives. Your answer should be close to one of the given choices.
- Eliminate alternatives you know are incorrect in each question. Then concentrate on the best alternative from the remaining answers.
- Take no more than five minutes if you are stuck on a multiple-choice question. Most multiple-choice questions should take about 1.5–2 minutes. If you are stuck, move on and come back to the question later.
- Do not change your answer unless you are sure and have a good reason.

3. The following are possible strategies for answering written-response questions.

- If possible, prepare an outline for your answer and use it as a guide when writing your final answer.
- When writing the answer, a rewriting of the question statement to start your answer, and a concluding statement to end your answer, is a good way to complete your response.
- Make sure your answer demonstrates clearly that you understand the concept, are organized in the presentation of your answer, and can effectively communicate your ideas.

4. No, because only 25% of the exam is of the recall (memorizing) type of question, and only a small portion of that would be specific to historical detail on mathematicians.

5. The written-response questions will require the most thoughtful and creative responses because you will be giving full, written answers.

6. If you get stuck on a multiple-choice or a numerical-response question, flag the question (mark it), leave it, and go on to the other questions. Come back to the question later. You may come across another question in the exam that will help you with the initial one.

7. There is no rule as to what order you complete the examination questions. Many students prefer to do the written response section first while they are fresh; others prefer to do the questions in the order given. Whatever order you choose, make sure you keep track of your time.

Section 3: Follow-up Activities

Extra Help

1. **a.** positive extreme area
   b. negative to positive z-score
   c. zero z-score to negative z-score
   d. positve extreme area and negative extreme area
   e. negative to negative z-score
   f. negative extreme area

2. **a.** $0 \le z \le 2.76$
   zero z-score to positive z-score

   b. $-2.82 \le z \le -1.09$
   negative to negative z-score

   c. $z \le -2.3$
   negative extreme area

   d. $z \le -2$ and $z \ge 2$
   positive extreme area and negative extreme area

   e. $2.3 \le z \le 2.32$
   positive to positive z-score

   f. $-0.04 \le z \le 1.03$
   negative to positive z-score

3. **a.** Area $= 0.3888$ for $z = 1.22$

   b. Area for $z = 2.76$ is 0.4971

   $$\text{Required area} = 0.5 - 0.4971$$
   $$= 0.0029$$

   c. Area for $z = 1.32$ is 0.4066

   $$\text{Total Area} = 0.4066 + 0.5$$
   $$= 0.9066$$

d. Area for $z = 1.58$ is 0.4429

Left Area $= 0.5 - 0.4429$
$= 0.0571$

Area for $z = 2.36$ is 0.4909

Right Area $= 0.5 - 0.4909$
$= 0.0091$

Total Area $= 0.0571 + 0.0091$
$= 0.0662$

4. a. Area $= 0.5 - 0.3461$
$= 0.1539$

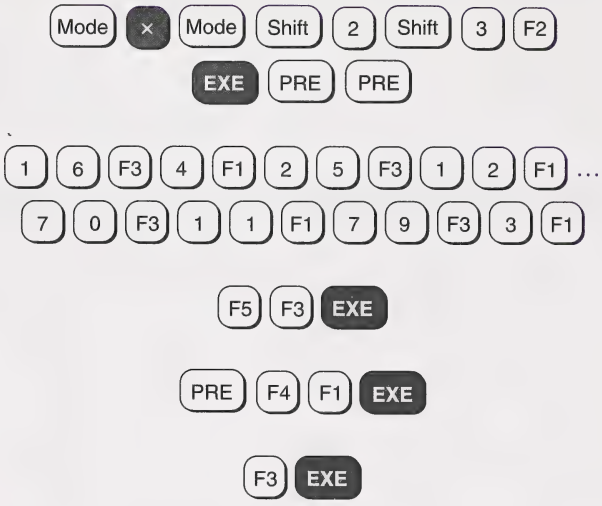
b. Area $= 0.5 + 0.4793$
$= 0.9793$

c. Area $= 0.4929$

d.

Enrichment

1.



The mean is approximately 46.0. The standard deviation is approximately 11.8.

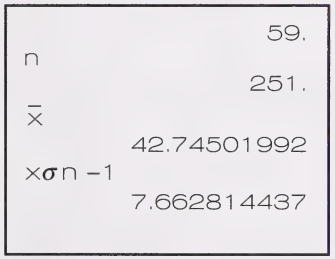
2. Mode × Mode Shift 2 Shift 3 F2
EXE PRE PRE

2 4 F3 8 F1 2 9 F3 1 0 F1 ...
5 4 F3 2 2 F1 5 9 F3 9 F1

F5 F3 EXE

PRE F4 F1 EXE

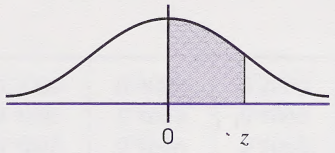
F3 EXE



The mean is approximately 42.7. The standard deviation is approximately 7.6.

Standard Normal Distribution Table

$$z = \frac{x - \mu}{\sigma}$$

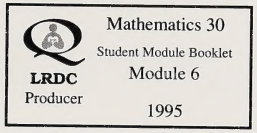


z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0. 1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319

(Table continues on next page.)

z	0	1	2	3	4	5	6	7	8	9
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998
3.5	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998	0.4998
3.6	0.4998	0.4998	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.7	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.8	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999	0.4999
3.9	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000

NLC/BNC
3 3286 50153 0592



Mathematics 30

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